Markov Chain Monte Carlo II

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Lecture plan

- MCMC Convergence
- Simulated Annealing
- Gibbs sampler

Convergence diagnostics

- Autocorrelation sequence $(x_1, ...x_n)$ k-th order aurocorrelation

$$\rho_k = \frac{\sum_{t=1}^{n-k}(x_t - \bar{x})(x_{t+k} - \bar{x})}{\sum_{t=1}^{n-k}(x_t - \bar{x})^2}, \quad \bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t$$

- Partial k-th autocorrelation as a function of lag
- Geweke z-score test
- split sample first 10%, last 50%
- at stationary means are equal, z-test. $z_{score} > 2$ still drifting

$$Z_{score} = \frac{\mu_1 - \mu_2}{\sigma}$$

Simulated annealing

- "Hill climbing"optimizaton algorithm
- accepts proabability of downhill move
- probability decreases with time (process)
- Metropolis sampling

$$\alpha(x_n, x^*) = min\left[1, \left(\frac{p(x^*)}{p(x_n)}\right)^{1/T(t)}\right]$$

- "Cooling"schedule

$$T(t) = T_0 \left(\frac{T_f}{T_0}\right)^{t/n}$$

- T_f - final "temperature"

The Gibbs sampler

- joint multivatiate density $p(\mathbf{x}) = p(x_1, ...x_n)$
- compute samples $\mathbf{X}_1, ..\mathbf{X}_k$
- compute marginal $p(x_1) = \int .. \int p(x_1..x_n)dx_1..dx_n$
- can compute univatiate $p(x_1|x_2...x_n)$
- generate sequence of univariate conditionals

Bivariate case

- distribution
  $p(x, y)$
- marginal distribution
  $p(x) = \int p(x, y)dy$
  $p(y) = \int p(x, y)dx$
- conditional probability
  $p(x|y) = p(x, y)/p(y)$
  $p(y|x) = p(x, y)/p(x)$
- marginal from conditional distribution
  $p(x) = \int p(x|y)p(y)dy = E_{p(y)}[p(x|y)]$
  $p(y) = \int p(y|x)p(x)dx = E_{p(x)}[p(y|x)]$

Gibbs sampler: bivariate case

- Given: $p(x|y), p(y|x)$
- Choose y_0, $t = 0$
- do "sampler scan"
  $x_t \sim p(x|y = y_t)$
  $y_{t+1} \sim p(y|x = x_t)$
- repeat k-times, Gibbs sequence $(x_0, y_0), (x_1, y_1)...(x_k, y_k)$

Gibbs sampler

- $p(x_1, ...x_n)$
- single iteration:
  $x_1^{t+1} = p(x_1|x_2^t....x_n^t)$
  $x_2^{t+1} = p(x_2|x_1^{t+1}, x_3^t....x_n^t)$
  $x_j^{t+1} = p(x_j|x_1^{t+1}, ..x_{j-1}^{t+1}, x_{j+1}^t, ..x_n^t)$
- samples
  $\mathbf{X}^1...\mathbf{X}^k$, $\mathbf{X} = (x_1, ..x_n)$

MCMC conection

- Metropolis-Hastings

$$\alpha(x, x^*) = \frac{p(x^*)Q(x^*, x)}{p(x)Q(x, x^*)} = \frac{p(x^*)Q(x|x^*)}{p(x)Q(x^*|x)}$$

- use conditional $p(x|x^*)$ as a candidate density $Q(x|x^*)$
  $(x^t, y^t) \to (x^{t+1}, y^t)$

$$\alpha = \frac{p(x^{t+1}, y^t)p(x^t|y^t)}{p(x^t, y^t)p(x^{t+1}|y^t)} = \frac{p(x^{t+1}, y^t)}{p(x^t, y^t)} \frac{p(x^t, y^t)}{p(y^t)} \frac{p(y^t)}{p(x^{t+1}, y^t)} = 1$$

- MCMC algorithm with acceptance probability 1