Graph Partitioning Algorithms

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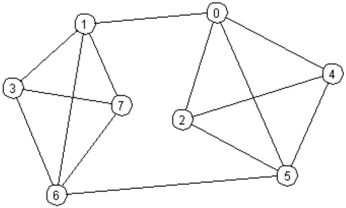
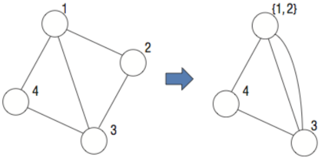
Minimum Cut

- *Graph cut* - partition vertices in two disjoint subsets.
- *Cut-set* of the cut is a set of edges with endpoints in defferent partitions.
- Cut size - the number of edges in cut-set
- *Min cut* is the smallest possible cut in the graph



Randomized Min Cut

Vertex contraction: replace conected $(v_1, v_2) \to v$, edges $E_v = E_{v_1} \cup E_{v_2}$



David Karger, 1993

Algorithm: Randomized Min-Cut

Input: Graph $G(V, E)$

Output: Graph min cut-set

repeat

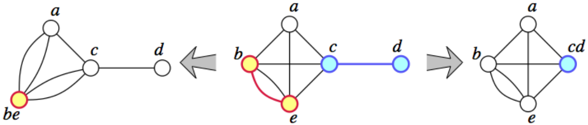
    pick a random edge e in G;

    contract its endpoints, $G' \leftarrow G \setminus e$;

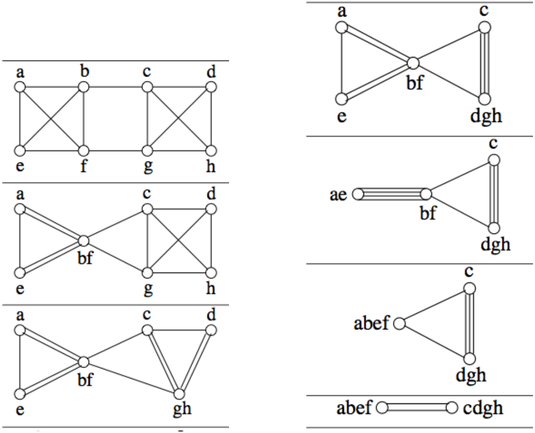
until *two vertices remain* ;

Randomized Min Cut

- For any cut in G' there is a cut in G with the same size (converse not true)
- Collapsing an edge can not decrease minimum cut size, $\min cut(G) \leq \min cut(G')$



Randomized Min Cut



Randomized Min Cut

Graph $G(m,n)$, m-edges, n-nodes
Let $\min(cut) = k$, then every node degree $k_i \geq k$, edges in graph $m \geq \frac{nk}{2}$
Probability randomly select an edge from the min cut $\frac{k}{m}$
Let E_i event that on i-th step selected edge is *not* in min cut.

$$
\begin{aligned}
P(E_1) &\geq 1 - \frac{k}{m} \geq 1 - \frac{2}{n} = \frac{n-2}{n} \\
P(E_2|E_1) &\geq 1 - \frac{2}{n-1} = \frac{n-3}{n-1} \\
P(E_i|E_1 \cup E_2 \cup ...E_{i-1}) &\geq 1 - \frac{2}{n-i+1} = \frac{n-i-1}{n-i+1}
\end{aligned}
$$

$$
P(E_1 \cup ..E_{n-2}) = P(E_1)P(E_2|E_1)....P(E_{n-2}|E_1 \cup E_2 \cup ...E_{n-3}) \geq
$$

$$
\geq \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \frac{n-5}{n-3} ... \frac{2}{4} = \prod_{i-1}^{n-2} \frac{n-i-1}{n-i+1} = \frac{2}{n(n-1)}
$$

Randomized Min Cut

- Probability of success - all selected edges are not in min cut (what's left is min cut)

$$P(success) \geq \frac{2}{n(n-1)}$$

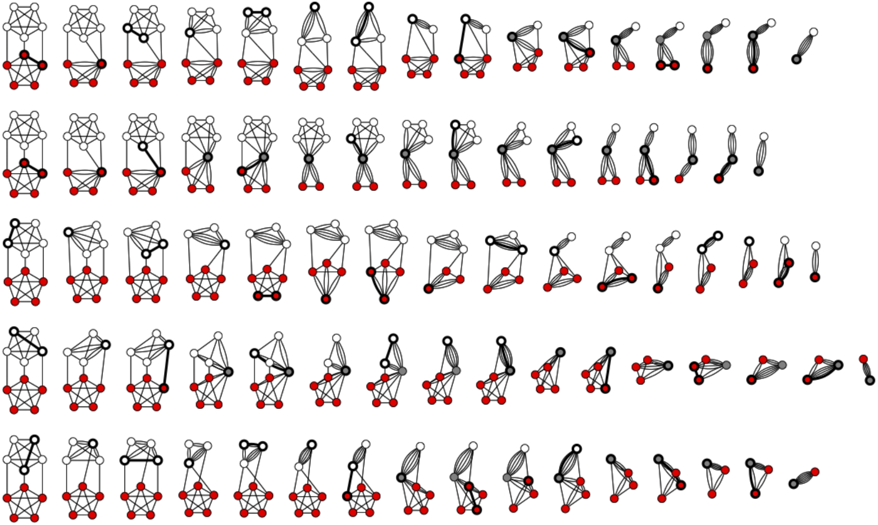
- Probability of *not* finding min cut after $N \sim n^2/2$ independent runs:

$$P(error) \leq \left(1 - \frac{2}{n(n-1)}\right)^{n^2/2} \sim \frac{1}{e} = 0.37$$

- With $N = c\frac{n(n-1)}{2} \log n$ independent runs

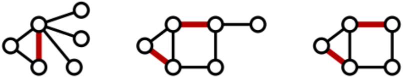
$$P(error) \leq \frac{1}{n^c}$$

Randomized Min Cut



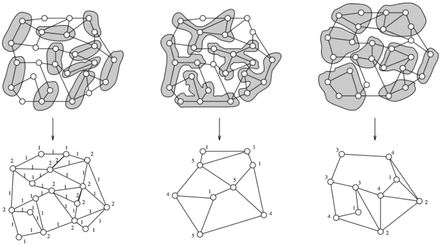
Graph Matching

- Matching - independent edge set, i.e set of edges without common veritces
- Maximal matching - if one more edge added, it is no longer a matching

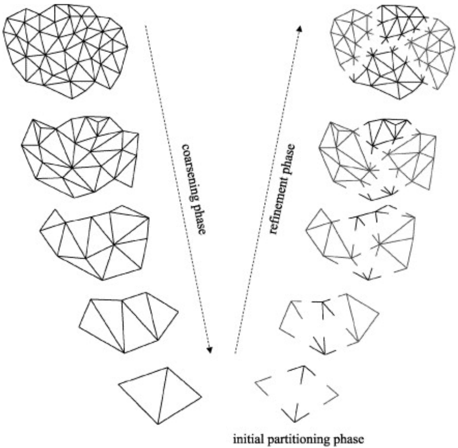


Coarsening schemes

- random matching, heavy-edge matching, light-edge matching
- multinode/hypergraph
- heavy clique matching (max density)



Multilevel Graph Partitioning



Multilevel Graph Partitioning

George Karypis, 1998

Algorithm: Multilevel graph partitioning

Input: Graph $G(V, E)$

Output: Graph partition

1. Coarsening: $G_0 \rightarrow G_1 \rightarrow G_2 ... \rightarrow G_m$, such that
$|V_0| > |V_1| > |V_2| > .. > |V_m|$
2. Partition: P_m
3. Uncoarsening: $P_m \rightarrow P_{m-1} \rightarrow P_{m-2} ... \rightarrow P_0$

- coarsening is done by randomized maximal matching
- partition on coarse graph can be done by greedy or advanced algorithms

References

- Global min-cuts in RNC, and other ramifications of a simple min-cut algorithm, D.R. Karger, SODA '93, pp 21-30, 1993
- Multilevel algorithms for partitioning power-law graphs, A. Abou-Rjeili, G. Karypis. IPDPS '06, p 124, 2006
- A fast and high quality multilevel scheme for partitioning irregular graphs. G.Karypis and V. Kumar, SIAM J. Sci. Comput. ,Vol. 20, No 1, pp. 359-392,1998