Power Laws

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Power Laws

- Node degrees $k_i = 1, 2, ...k_{\max}$
- Degree distribution $P(k) \equiv P(k_i = k)$

$$P(k) = \frac{n_k}{n} = \frac{n_k}{\sum_k n_k}$$

- Power law

$$P(k) = ck^{-\gamma} = \frac{c}{k^{\gamma}}$$

- Logarithmic coordinates

$$\log(P(k)) = -\gamma \log k + \log c$$

Power Laws

Continues approximation $P(x) = Cx^{-\alpha}$

- Normalization $(\alpha > 1)$

$$\int_{x_{min}}^{\infty} P(x)dx = \frac{c}{\alpha - 1} x_{\text{min}}^{-\alpha+1} = 1$$

$$C = (\alpha - 1) x_{\text{min}}^{\alpha - 1}$$

- mean $(\alpha > 2)$

$$\langle x \rangle = \int_{x_{\text{min}}}^{\infty} xP(x)dx = \frac{\alpha - 1}{\alpha - 2} x_{\text{min}}$$

- deviation $(\alpha > 3)$

$$\langle x^2 \rangle = \int_{x_{\text{min}}}^{\infty} x^2 P(x)dx = \frac{\alpha - 1}{\alpha - 3} x_{\text{min}}^2$$

Power Laws

- CDF

$$\Pi(x) = \int_{-\infty}^{x} p(\xi)d\xi = \int_{x_{\min}}^{x} p(\xi)d\xi$$

- cCDF

$$F(x) = 1 - \Pi(x) = \int_{x}^{\infty} p(\xi)d\xi$$

- for $p(x) = Cx^{-\alpha}$

$$F(x) = \frac{C}{\alpha - 1} x^{-(\alpha-1)} = \left(\frac{x}{x_{\min}}\right)^{-(\alpha-1)}$$

Power Laws

Discrete case $P(k) \equiv P(k_i = k)$, $k = 1, 2, ...k_{\max}$

- power distribution

$$p_k = ck^{-\gamma} = \frac{c}{k^{\gamma}}$$

- normalization

$$\sum_{k=1}^{\infty} p_k = c \sum_{k=1}^{\infty} k^{-\gamma} = c\zeta(\gamma) = 1$$

- Riemann zeta function, $\gamma > 1$

$$p_k = \frac{k^{-\gamma}}{\zeta(\gamma)}$$

Power Laws

Maximum likelihood estimation of parameter α

- Let $\{x_i\}$ be a set of n observations (points) independetly sampled from the distribution

$$P(x_i) = \frac{\alpha - 1}{x_{\min}} \left(\frac{x_i}{x_{\min}} \right)^{-\alpha}$$

- Probability of the sample

$$P(\{x_i\}|\alpha) = \prod_i^n \frac{\alpha - 1}{x_{\min}} \left(\frac{x_i}{x_{\min}} \right)^{-\alpha}$$

- Bayes' theorem

$$P(\alpha|\{x_i\}) = P(\{x_i\}|\alpha) \frac{P(\alpha)}{P(\{x_i\})}$$

Power Laws

- log-likelihood

$$\mathcal{L} = \ln P(\alpha|\{x_i\}) = n\ln(\alpha - 1) - n\ln x_{\min} - \alpha \sum_{i=1}^{n} \ln \frac{x_i}{x_{\min}}$$

- maximization $\frac{\partial \mathcal{L}}{\partial \alpha} = 0$

$$\alpha = 1 + n\left[\sum_{i=1}^{n} \ln \frac{x_i}{x_{\min}}\right]^{-1}$$

- error estimate

$$\sigma = \sqrt{n}\left[\sum_{i=1}^{n} \ln \frac{x_i}{x_{\min}}\right]^{-1} = \frac{\alpha - 1}{\sqrt{n}}$$

References

- Power laws, Pareto distributions and Zipf's law, M. E. J. Newman, Contemporary Physics, pages 323–351, 2005.