Random Graphs

Leonid E. Zhukov

School of Applied Mathematics and Information Science
National Research University Higher School of Economics

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НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ
УНИВЕРСИТЕТ

Random Graph models

Graph $G\{E, V\}$, nodes $n = |V|$, edges $m = |E|$
Erdos and Renyi, 1959.
Random graph models

- $G_{n,m}$, a randomly selected graph from the set of $C^m_{n(n-1)/2}$ graphs with n nodes and m edges
- $G_{n,p}$, each pair from n nodes is connected with probability p, m - random number
- A graph with m edges have the probability

$$P(G_{n,p}(m)) = p^m (1-p)^{n(n-1)/2-m}$$

$$\langle m \rangle = p\frac{n(n-1)}{2}$$

$$\langle k \rangle = \frac{1}{n}\sum_i k_i = \frac{2m}{n} = p\ (n-1) \approx pn$$

Random Graph models

- Bernoulli distribution

$$P(k_i = k) = P(k) = C_{n-1}^{k} p^k (1-p)^{n-1-k}$$

- Poisson distribution

$$P(k) = \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!} = \frac{z^k e^{-z}}{k!}$$

n the limit $n \to \infty$ at fixed $\langle k \rangle = pn = z$

Phase transition

Consider $G_{n,p}$ as a function of p

- $p = 0$, empty graph
- $p = 1$, complete (full) graph
- There are exist critical p_c, structural changes from $p < p_c$ to $p > p_c$
- Gigantic connected component appears at $p > p_c$

Phase transition

Let u – fraction of nodes that do not belong to GCC. The probability that a node does not belong to GCC

$$u = P(k=1)\cdot u + P(k=2)\cdot u^2 + P(k=3)\cdot u^3 ... =$$
$$= \sum_{k=0}^{\infty} P(k) u^k = \sum_{k=0} \frac{z^k e^{-z}}{k!} u^k = e^{-z} e^{zu} = e^{z(u-1)}$$

$$s = 1 - u$$

$$1 - s = e^{-zs}$$

when $z \to \infty, \quad s \to 1$
when $z \to 0, \quad s \to 0$

Let s -fraction of nodes belonging to GCC

$$1 - s = e^{-zs}$$

non-zero solution appears when

$$1 = ze^{-zs}$$

critical value

$$z = 1, \quad z = p_c n,$$

$$\langle k \rangle = p_c n = 1, \quad p_c = \frac{1}{n}$$

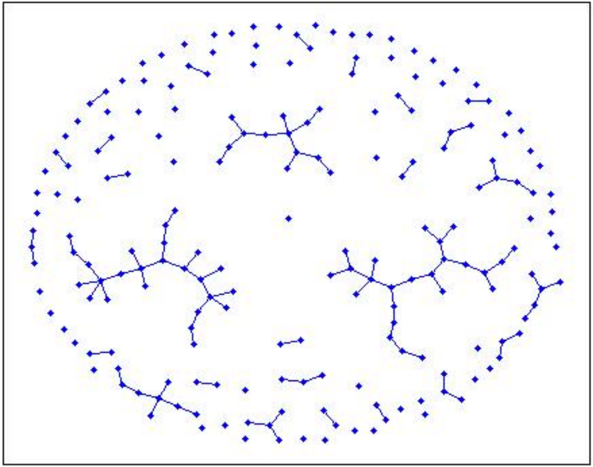
on average a node has more than one neighbour

Phase transition

for $n \to \infty$

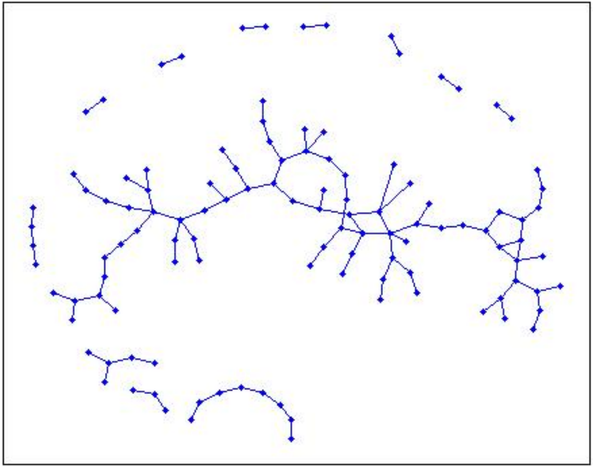
- when $p < p_c = 1/n$, $(\langle k \rangle < 1)$ there is no components with more than $O(\ln n)$ nodes, all components are either trees or have maximum one cycle, largest component is a tree
- when $p = p_c = 1/n$, $(\langle k \rangle = 1)$ the largest component has $O(n^{2/3})$ nodes
- when $p > p_c = 1/n$, $(\langle k \rangle > 1)$ gigantic component has all $O(n)$ nodes

Phase transition



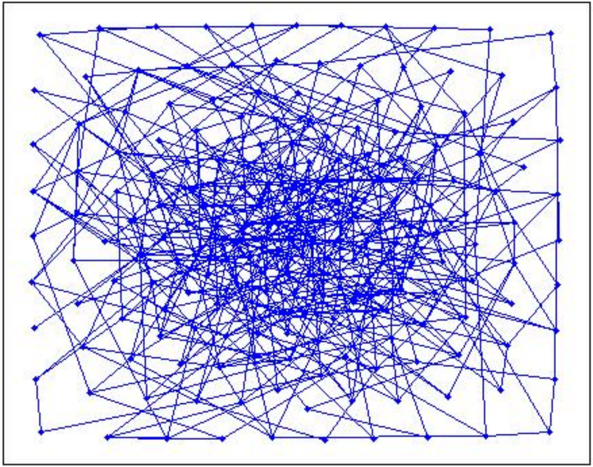
$p < p_c$

Phase transition



$p = p_c$

Phase transition



$p > p_c$

Random Graphs

- Graph diameter when $p > p_c$ $\langle k \rangle^d = n$

$$d = \frac{\ln n}{\ln \langle k \rangle} = \frac{\ln n}{\ln pn}$$

- Clustering coefficient

$$C(k) = \frac{\#\text{of links between NN}}{\#\text{max number of links NN}} = \frac{pk(k-1)/2}{k(k-1)/2} = p$$

$$C = p = \frac{\langle k \rangle}{n}$$

when $n \to \infty, \quad C \to 0$

Configuration model

Select a sequence of nodes with degreees
$D = \{k_1, k_2, k_3..k_n\} : \sum_i k_i = 2m$ to follow given distribution $P(k)$. For example: 1 1 1 1 1 2 2 2 3 3 3...

$$P(k) = \frac{\#(k_i = k)}{2m}$$

Randomly select two nodes from the sequence and form an edge between them

Refernces

- On random graphs I, P. Erdos and A. Renyi, Publicationes Mathematicae 6, 290–297 (1959).
- On the evolution of random graphs, P. Erdos and A. Renyi, Publicaton of the Mathematical Institute of the Hungarian Academy of Sciences, 17-61 (1960)