Epidemic models

Leonid E. Zhukov

School of Applied Mathematics and Information Science
National Research University Higher School of Economics

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Epidemic dynamics models

- W. O. Kermack and A. G. McKendrick, 1927
- Deterministic compartamental model (population classes) $\{S, I, T\}$
- $S(t)$ - succeptable, number of individuals not yet infected with the disease at time t $I(t)$ - infected, number of individuals who have been infected with the disease and are capable of spreading the disease.
- $R(t)$ - recoverd, number of individuals who have been infected and then recovered from the disease, can't be infected again or to transmit the infection to others.
- Fully-mixing model
- Cosed population (no birth, death, migration)
- Models: SI, SIS, SIR, SIRS,..

SI model

- $S(t)$ -susceptable , $I(t)$ - infected

$$S \longrightarrow I$$

- β - infection rate (on contact)

$$S(t) + I(t) = N$$

- Infection equation:

$$I(t + \delta t) = I(t) + \beta \frac{S(t)}{N} I(t) \delta t$$
$$\frac{dI(t)}{dt} = \beta \frac{S(t)}{N} I(t)$$

SI model

- Normalization: $i(t) = I(t)/N$, $s(t) = S(t)/N$
- Equations

$$\begin{aligned} \frac{ds(t)}{dt} &= -\beta s(t)i(t) \\ \frac{di(t)}{dt} &= \beta s(t)i(t) \end{aligned}$$

$$s + t = 1$$

SI model

- Differential equation, $i(t=0)=i_0$

$$\frac{di(t)}{dt} = \beta(1-i(t))i(t)$$

- Solution:

$$\int \frac{di}{(1-i)i} = \beta \int dt + C$$

$$\log \frac{i(t)}{1-i(t)} = \beta t + C$$

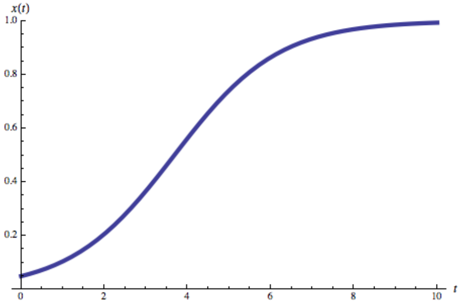
- Logistic growth function:

$$i(t) = \frac{i_0}{i_0 + (1-i_0)e^{-\beta t}}$$

- Population:

$$\begin{aligned} I(t) &= i(t)N \\ S(t) &= (1-i(t))N \end{aligned}$$

Logistic growth function

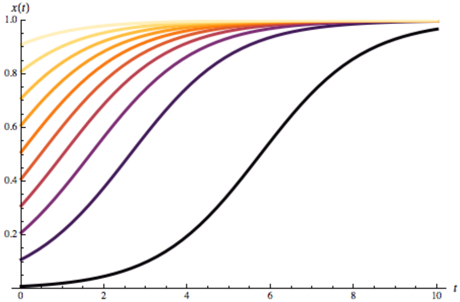


- $i_0 = 0.05, \quad \beta = 0.8$
- $t \to \infty$

$$I(\infty) = N$$

$$S(\infty) = 0$$

Logistic growth function



- $\beta = 0.8$

SIS model

- $S(t)$ -susceptable , $I(t)$ - infected,

$$S \longrightarrow I \longrightarrow S$$

- β - infection rate (on contact), γ - recovery rate

$$S(t) + I(t) = N$$

- Infection equation:

$$\begin{aligned} \frac{ds}{dt} &= -\beta si + \gamma i \\ \frac{di}{dt} &= \beta si - \gamma i \end{aligned}$$

$$s + i = 1$$

- $$\frac{di}{dt} = (\beta - \gamma - i)i$$

SIS model

- Solution

$$i(t) = (1 - \frac{\gamma}{\beta})\frac{C}{C + e^{-(\beta-\gamma)t}}, \qquad C = \frac{\beta i_0}{\beta - \gamma - \beta i_0}$$

- Limit $t \to \infty$

$$\beta > \gamma \quad , \qquad i(t) \to (1 - \frac{\gamma}{\beta})$$

$$\beta < \gamma \quad , \qquad i(t) = i_0 e^{(\beta-\gamma)t} \to 0$$

- $\beta > \gamma$

$$I(\infty) = N(1 - \frac{\gamma}{\beta})$$

$$S(\infty) = N\frac{\gamma}{\beta}$$

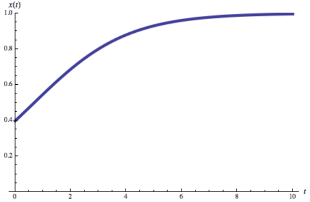
- $\beta < \gamma$

$$I(\infty) = 0$$

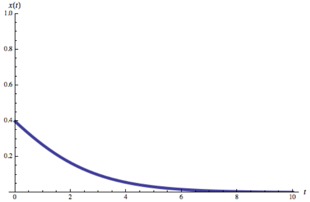
$$S(\infty) = N$$

Logistic function

- $\beta > \gamma, \quad i(t) \to (1 - \frac{\gamma}{\beta})$



- $\beta < \gamma, \quad i(t) = i_0 e^{(\beta-\gamma)t} \to 0$



SIR model

- $S(t)$ -susceptable , $I(t)$ - infected, $R(t)$ - recovered

$$S \longrightarrow I \longrightarrow R$$

- β - infection rate, γ - recovery rate

$$S(t) + I(t) + R(t) = N$$

- Infection equation:

$$\begin{aligned}
\frac{ds}{dt} &= -\beta si \\
\frac{di}{dt} &= \beta si - \gamma i \\
\frac{dr}{dr} &= \gamma i
\end{aligned}$$

$$s + i + r = 1$$

SIR model

- Solution

$$\frac{ds}{dt} = -\beta s \frac{dr}{dt}\frac{1}{\gamma}$$

$$s = s_0 e^{-\frac{\beta}{\gamma} r}$$

$$\frac{dr}{dt} = \gamma i = \gamma(1 - s - r) = \gamma(1 - r - s_0 e^{-\frac{\beta}{\gamma} r})$$

- Limits: $t \to \infty, \quad r(\infty) \to const, \quad \frac{dr}{dt} = 0$

$$1 - r = s_0 e^{-\frac{\beta}{\gamma} r}$$

- Initial conditions: $r(0) = 0$, $i(0) = c/N$, $s(0) = 1 - c/N \approx 1$

$$1 - r = e^{-\frac{\beta}{\gamma} r}$$

SIR model

- Critical point

$$(1 - r)' = (e^{-\frac{\beta}{\gamma} r})'$$

$$\frac{\beta}{\gamma} = 1$$

- Epidemic threshold

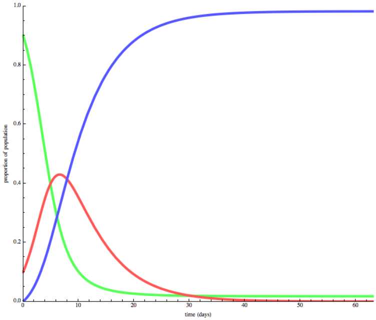
$$\beta > \gamma \quad , \quad r(\infty) = const > 0$$
$$\beta < \gamma \quad , \quad r(\infty) \to 0$$

- Basic reproduction number

$$R_0 = \frac{\beta}{\gamma}$$

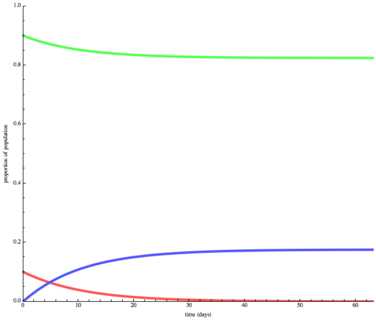
- $R_0 > 1$ - epidemics
  $R_0 < 1$ - no epidemics

SIR model



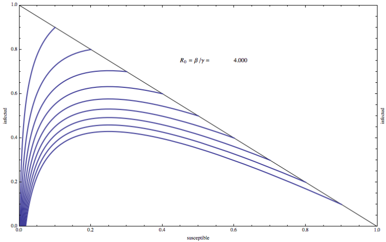
- $R_0 = 4$
- $i_0 = 0.1$

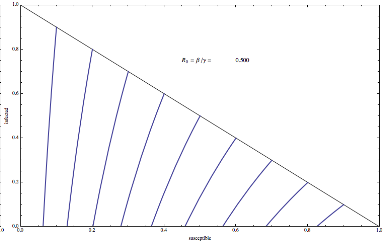
SIR model



- $R_0 = 0.5$
- $i_0 = 0.1$

Phase diagram





References

- A Contribution to the Mathematical Theory of Epidemics. , Kermack, W. O. and McKendrick, A. G. , Proc. Roy. Soc. Lond. A 115, 700-721, 1927.
- The Mathematics of Infectious Disease, Herbert W. Hethcote, SIAM Review, Vol. 42, No. 4, p. 599-653, 2000