Epidemics on networks

Leonid E. Zhukov

School of Applied Mathematics and Information Science
National Research University Higher School of Economics

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НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ
УНИВЕРСИТЕТ

Epidemic models on networks

- network of potential contacts, adjacency matrix **A**
- probabilistic model:
  $s_i(t)$ - probability that at t node i is susceptable
  $x_i(t)$ - probability that at t node i is infected
  $r_i(t)$ - probability that at t node i is recovered
- network properties:
  - averege number of NN, $\langle k \rangle$, degree distribution $p(k)$
  - average path length $\langle l \rangle$, distribution A^m
  - nodes reachability: connected components
  - triangles: clustering coefficient C

SI model

- $s_i(t)$ - susceptable, $x_i(t)$ -infected

$$S \longrightarrow I$$

$$x_i(t) + s_i(t) = 1$$

- β - probability to get infected

$$x_i(t + \delta t) = x_i(t) + \beta s_i \sum_i A_{ij} x_j \delta t$$

- infection equation

$$\begin{aligned} \frac{dx_i(t)}{dt} &= \beta s_i(t) \sum_j A_{ij} x_j(t) \\ \frac{ds_i(t)}{dt} &= -\beta s_i(t) \sum_j A_{ij} x_j(t) \end{aligned}$$

SI model

- Differential equation

$$\frac{dx_i(t)}{dt} = \beta(1 - x_i(t)) \sum_j A_{ij} x_j$$

- "early time"approximation, $x_i(t) \ll 1$

$$\frac{dx_i(t)}{dt} = \beta \sum_j A_{ij} x_j$$

$$\frac{d\mathbf{x}(t)}{dt} = \beta \mathbf{A}\mathbf{x}(t)$$

- Solution in the basis

$$\mathbf{A}\mathbf{v}_k = \lambda_k \mathbf{v}_k$$

$$\mathbf{x}(t) = \sum_k a_k(t) \mathbf{v}_k$$

SI model

$$\sum_k \frac{da_k}{dt}\mathbf{v}_k = \beta \sum_k \mathbf{A} a_k(t)\mathbf{v}_k = \beta \sum_k a_k(t)\lambda_k \mathbf{v}_k$$

$$\frac{da_k(t)}{dt} = \beta\lambda_k a_k(t)$$

$$a_k(t) = a_k(0)e^{\beta\lambda_k t}$$

- Solution

$$\mathbf{x}(t) = \sum_k a_k(0)e^{\lambda_k \beta t}\mathbf{v}_k$$

- $t \to 0$, $\lambda_{max} = \lambda_1 > \lambda_k$

$$\mathbf{x}(t) = \mathbf{v}_1 e^{\lambda_1 \beta t}$$

1. growth rate of infections depends on λ_1
2. probability of infection of nodes depends on $\mathbf{v}_1$, i.e v_{1i}

SI model

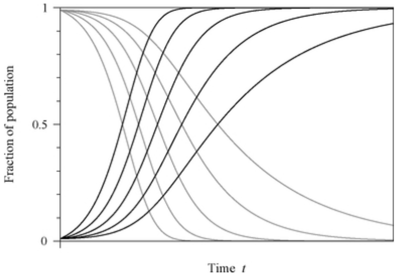
- long time, $x_i(t) \to const$

$$\frac{dx_i(t)}{dt} = \beta(1 - x_i(t)) \sum_j A_{ij} x_j = 0$$

  $\mathbf{Ax} \neq 0$ since $\lambda_{min} \neq 0$, $1 - x_i(t) \approx 0$

- $t \to \infty \quad x_i(t) \to 1$

SI model



SIS model

- $s_i(t)$ -susceptable , $x_i(t)$ - infected,

$$S \longrightarrow I \longrightarrow S$$

- β - infection rate, γ - recovery rate

$$x_i(t) + s_i(t) = 1$$

$$\begin{aligned}
\frac{dx_i(t)}{dt} &= \beta s_i(t) \sum_j A_{ij} x_j(t) - \gamma x_i \\
\frac{ds_i(t)}{dt} &= -\beta s_i(t) \sum_j A_{ij} x_j(t) + \gamma x_i
\end{aligned}$$

SIS model

- Differential equation, $i(t=0) = i_0$

$$\frac{dx_i(t)}{dt} = \beta(1 - x_i(t)) \sum_j A_{ij} x_j - \gamma x_i$$

- "early time"approximation, $x_i(t) \ll 1$

$$\frac{dx_i(t)}{dt} = \beta \sum_j A_{ij} x_j - \gamma x_i$$

$$\frac{dx_i(t)}{dt} = \beta \sum_j (A_{ij} - \frac{\gamma}{\beta}\delta_{ij}) x_j$$

$$\frac{d\mathbf{x}(t)}{dt} = \beta \mathbf{M}\mathbf{x}(t), \quad \mathbf{M} = \mathbf{A} - (\frac{\gamma}{\beta})\mathbf{I}$$

SIS model

- Solution in the basis

$$\begin{aligned} \mathbf{M}\mathbf{v}'_k &= \lambda'_k \mathbf{v}'_k, \quad \mathbf{M} = \mathbf{A} - (\frac{\gamma}{\beta})\mathbf{I} \\ \mathbf{v}'_k &= \mathbf{v}_k, \quad \lambda'_k = \lambda_k - \frac{\gamma}{\beta} \\ \mathbf{x}(t) &= \sum_k a_k(t)\mathbf{v}_k = \sum_k a_k(0)\mathbf{v}_k e^{\lambda'_k \beta t} = \sum_k a_k(0)\mathbf{v}_k e^{(\beta\lambda_k - \gamma)t} \end{aligned}$$

- $\lambda_{max} = \lambda_1 \geq \lambda_k$
  if $\beta\lambda_1 > \gamma$, $\mathbf{x}(t) \to \mathbf{v}_1 e^{(\beta\lambda_1 - \gamma)t}$, growth
  if $\beta\lambda_1 < \gamma$, $\mathbf{x}(t) \to 0$, decay
  critical: $\beta\lambda_1 = \gamma$
- Epidemic threshold

$$R_0 = \frac{\beta}{\gamma} = \frac{1}{\lambda_1}$$

small λ_1 - difficult for epidemics to spread
large λ_1 - easy for epidemics to spread

SIS model

- long time, $x_i(t) \to const$

$$\frac{dx_i(t)}{dt} = \beta(1 - x_i(t)) \sum_j A_{ij} x_j - \gamma x_i = 0$$

$$x_i = \frac{\sum_j A_{ij} x_j}{\frac{\gamma}{\beta} + \sum_j A_{ij} x_j}$$

- if $\gamma \ll \beta$, $x_i(t \to \infty) = 1$
  if $\gamma \sim \beta$, $x_i \frac{\gamma}{\beta} = \sum_j A_{ij} x_j$
  $\lambda_1 = \frac{\gamma}{\beta}, \quad x_i(t \to \infty) = v_i$

SIR model

- $s_i(t)$ -susceptable , $x_i(t)$ - infected, $r_i(t)$ - recovered

$$S \longrightarrow I \longrightarrow R$$

- β - infection rate, γ - recovery rate

$$s_i(t) + x_i(t) + r_i(t) = 1$$

- Infection equation:

$$\begin{aligned} \frac{ds_i}{dt} &= -\beta s_i \sum_j A_{ij} x_j \\ \frac{dx_i}{dt} &= \beta s_i \sum_j A_{ij} x_j - \gamma x_i \\ \frac{dr_i}{dt} &= \gamma x_i \end{aligned}$$

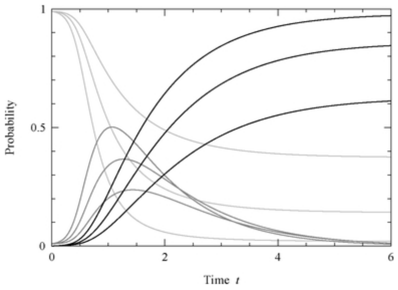
SIR model

- early time, $t \to 0$, $r_i \sim 0$

$$\frac{dx_i(t)}{dt} = \beta(1 - r_i - x_i)\sum_j A_{ij}x_j - \gamma x_i$$

SIS = SIR model at early times, $\mathbf{x}(t) \sim \mathbf{v}_1 e^{(\beta\lambda_1 - \gamma)t}$

SIR model



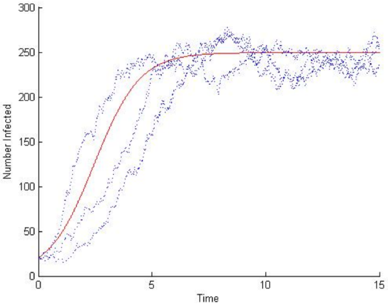
Modeling SIS

1. Every node at any time step is in one state $\{S, I\}$
2. Initialize c nodes in state I
3. Each node stay infected $\tau_\gamma = 1/\gamma$ time steps
4. On each time step each I node has a prabability β to infect its nearest neighbours (NN), $S \to I$
5. After τ_γ time steps node recovers, $I \to S$

Modeling SIR

1. Every node at any time step is in one state $\{S, I, R\}$
2. Initialize c nodes in state I
3. Each node stay infected $\tau_\gamma = 1/\gamma$ time steps
4. On each time step each I node has a prabability β to infect its nearest neighbours (NN), $S \to I$
5. After τ_γ time steps node recovers, $I \to R$
6. Nodes R do not participate in infection propagation

Modeling



References

- Networks and Epidemics models Matt. J. Keeling and Ken.T.D. Eames, J. R. Soc. Interfac, 2, 295-307, 2005