# Threshold models and Influence maximization


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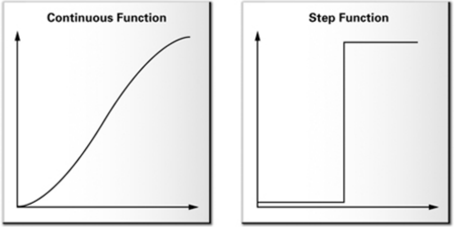
17.05.2013

# Threshold models

Threshold models of Collective Behavior. Mark Granovettor, 1978

- Adoption of innovation, voting, applause, leaving social occasion, riots
- Group of people, each to make a decision
- Binary mutually exclusive decision: adopt/reject, stay/go, join/not join
- Every person has own preference, decision threshold
- Costs and benefits depends on how many others make which choice
- Dynamical proces with equilibrium outcome (final proportion of making each decision)
- Example: insitgator + crowd
  $5, 5, 5, 5, 5, 5, 5, 5, 5$
  $1, 2, 3, 4, 5, 6, 7, 8, 9$ (domino effect)
  $2, 3, 4, 5, 6, 7, 8, 9$

# Threshold function



# Threshold models

- Let $i's$ threshold level $\theta(i)$, $x$ - number of participans
- if $x \geq \theta(i)$ - join, $x < \theta(i)$ - not joint
- Let $f(x)$ - number of people with thresholod level $\theta = x$
  $F(x)$ - number of people with $\theta \leq x$ (cumulative function)

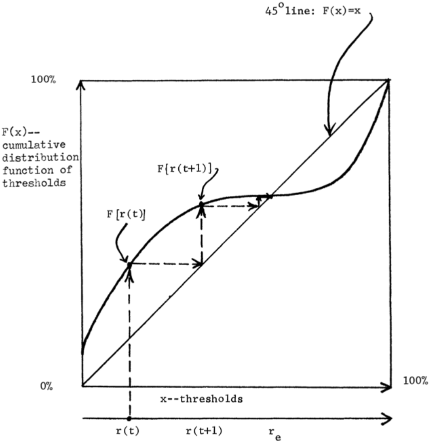
$$F(x) = \sum_{x'}^{x} f(x')$$

- Initital state $x_0$

$$x_1 = F(x_0)$$
$$x_2 = F(x_1)$$
$$x_{t+1} = F(x_t)$$

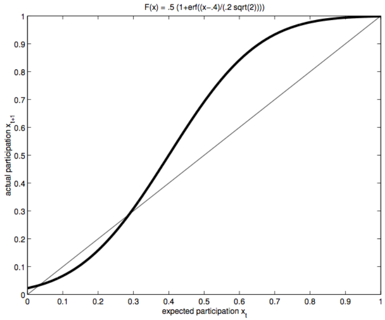
- Fixed point

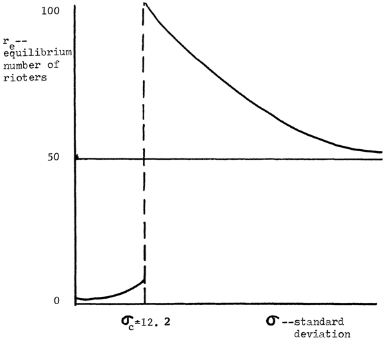
$$x^* = F(x^*)$$


45°line: F(x)=x
100%
F(x)--
cumulative
distribution
function of
thresholds
F[r(t+1)]
F[r(t)]
0%
100%
x--thresholds
r(t)
r(t+1)
r
e

$y = x$
$y = F(x)$

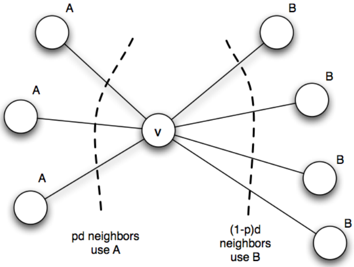
# Granovetter model



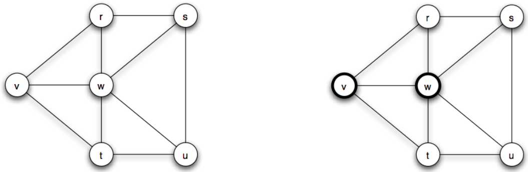

$$y = x$$
$$y = 1/2(1 + erf(\frac{x-\mu}{\sigma\sqrt{2}}))$$

# Network model



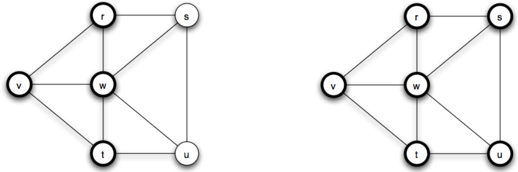

Node $v$ to make decision $A$ or $B$
$p$ - portion of type $A$ neighbors
To accept $A$: $a \cdot p \cdot d > b \cdot (1 - p) \cdot d$
$p > b/(a + b)$

# Cascades




(a) *The underlying network*

(b) *Two nodes are the initial adopters*




(c) *After one step, two more nodes have adopted*

(d) *After a second step, everyone has adopted*

$a = 3, b = 2$, threshold $p > 2/5$

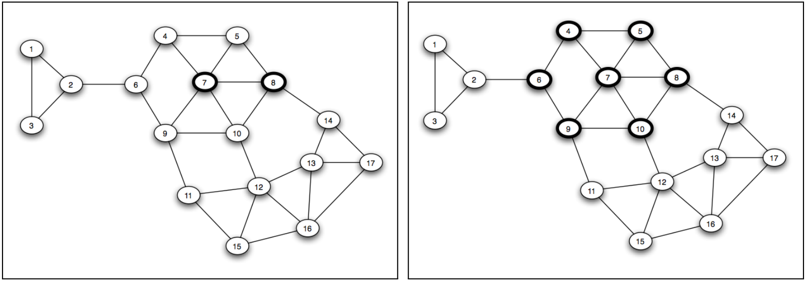
# Linear threshold model

- Influence comes only from NN $N(i)$ nodes, $w_{ij}$ influence $i \to j$
- Require $\sum_{j \in N(i)} w_{ji} \leq 1$
- Each node has a random acceptance threshold from $\theta_i \in [0, 1]$
- Activation: fraction of active nodes exceeds threshold

$$\sum_{active\ \ j \in N(i)} w_{ji} > \theta_i$$

- Initial set of active nodes $A_o$, iterative process with discrete time steps
- Progressive process, only nonactive $\to$ *active*

# Cascades




- Inital set of active nodes $A_o$
- Cascade size $\sigma(A_o)$ - number of active nodes when propagation stops
- Find $k$-set of nodes $A_o$ that produces maximal cascade $\sigma(A_o)$
- $k$-set of "maximum influence"nodes
- NP-hard

# Submodular functions

- Set function $f$ is submodular, if for sets $S$, $T$ and $S \subseteq T$, $\forall v \notin T$

$$f(S \cup \{v\}) - f(S) \geq f(T \cup \{v\}) - f(T)$$

- Function of diminishing returns
- Function $f$ is monotone, $f(S \cup \{v\}) \geq f(S)$

**Theorem**

*Let $F$ be a monotone submodular function and let $S^*$ be the k-element set achieving maximal $f$. Let $S$ be a k-element set obtained by repeatedly, for k-iterations, including an element producing the largest marginal increase in $f$.*

$$f(S) \geq (1 - \frac{1}{e}) f(S^*)$$

Nemhauser, Wolsey, and Fisher, 1978

# Influence maximization

- $\sigma(.)$- monotone increasing function
- $\sigma(.)$- submodular function (D. Kempe, J. Kleinberg, E. Tardos, 1993)

# Influence maximization

Approximation algorithm

**Algorithm:** Greedy optimization

**Input**: Graph $G(V, E)$, k

**Output**: Maxumum influence set $S$

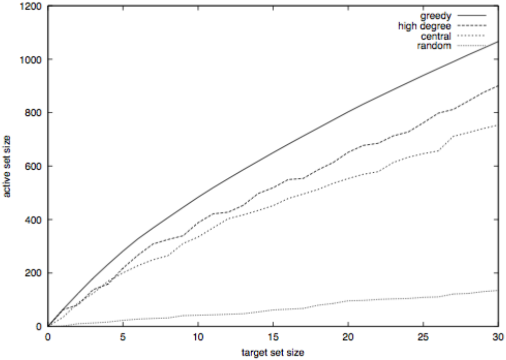
Set $S \leftarrow 0$

**for** $i = 1 : k$ **do**

  select $v = \arg\max_{u \in V \setminus S}(\sigma(S \cup \{u\}) - \sigma(S))$

  $S \leftarrow S \cup \{v\}$

# Linear threshold model




network: collaboration graph
10,000 nodes, 53,000 edges

# References


- Threshold Models of Collective Behavior Mark S. Granovetter, American Journal of Sociology 83(6):1420-1443, 1978.
- Maximizing the Spread of Influence through a Social Network, D. Kempe, J. Kleinberg, E. Tardos, 2003
- Influential Nodes in a Diffusion Model for Social Networks, D. Kempe, J. Kleinberg, E. Tardos