

Social Influence. Reaching Consensus

Leonid E. Zhukov

School of Applied Mathematics and Information Science
National Research University Higher School of Economics

24.05.2013



НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ
УНИВЕРСИТЕТ

Reaching a Consensus, Morris DeGroot 1974

Consensus = mutual agreement on a subject among group of people

- Group of people with opinions on the subject
- Can a common belief be reached?
- How long would it take?
- How each individual belief contribute to consensus?
- Which individuals have the most influence over final beliefs?

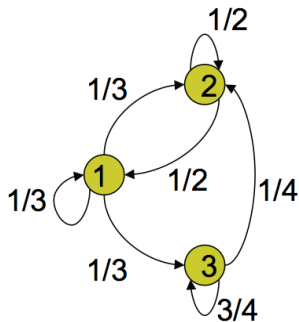
- Opinion $p_i(t) \in [0..1]$,
- T_{ij} is weight on the opinion of others, $i \rightarrow j$, how much i "listens to opinion" of j
- $\sum_j T_{ij} = 1$, row-stochastic matrix
- Opinion update

$$p_i(t+1) = \sum_j T_{ij} p_j(t)$$

- Consensus is reached if

$$\exists \lim_{t \rightarrow \infty} p_i(t) = p^\infty$$

Example 1



$$T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

Example 1

- Initial beliefs

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Updating

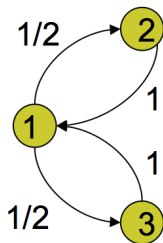
$$p(1) = Tp(0) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix}$$

$$p(2) = Tp(1) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5/18 \\ 5/12 \\ 1/8 \end{pmatrix}$$

- Consensus

$$p(t) = Tp(t-1) = T^t p(0) \rightarrow \begin{pmatrix} 3/11 \\ 3/11 \\ 3/11 \end{pmatrix}$$

Example 2



$$T = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Example 2

- Initial beliefs

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Updating

$$p(1) = Tp(0) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$p(2) = Tp(1) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Consensus

$$\nexists \lim_{t \rightarrow \infty} T^t p(0)$$

Perron - Frobenius Theorem

Linear algebra, Graph theory, Markov chain theory.

Perron & Frobenius, 1912

Theorem

Let A be a square non-negative $A_{ij} \geq 0$ irreducible matrix and $\rho(A) = \max_i (|\lambda_i|)$ its spectral radius. Then

1. $\rho(A) > 0$ and $\lambda_\rho = \rho(A)$ is an eigenvalue with multiplicity one
2. Left and right eigenvectors associated with λ_ρ are strictly positive
3. Any other eigenvalue $|\lambda| < \rho(A)$
4. $\min_i \sum_j A_{ij} \leq \rho(A) \leq \max_i \sum_j A_{ij}$

Reducible matrix:

$$P^T A P = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

Irreducible matrix = strongly connected graph

• Example 1:

$$T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

Right: $\mathbf{T}\mathbf{p} = \lambda\mathbf{p}$, $\lambda = \{1, 0.5, 0.083\}$, $\rho_1 = \begin{pmatrix} 0.58 \\ 0.58 \\ 0.58 \end{pmatrix}$

Left: $\mathbf{p}\mathbf{T} = \mathbf{p}\lambda$, $\lambda = \{1, 0.5, 0.083\}$, $\rho_1 = (0.46, 0.62, 0.62)$

• Example 2:

$$T = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

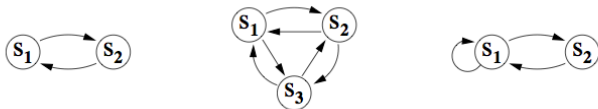
Right: $\mathbf{T}\mathbf{p} = \lambda\mathbf{p}$, $\lambda = \{1, 0, -1\}$, $\rho_1 = \begin{pmatrix} 0.58 \\ 0.58 \\ 0.58 \end{pmatrix}$

Left: $\mathbf{p}\mathbf{T} = \mathbf{p}\lambda$, $\lambda = \{1, 0, -1\}$, $\rho_1 = (0.82, 0.41, 0.41)$

- **T** - row-stochastic (right stochastic) matrix , i.e. $T_{ij} \geq 0$, $\sum_j T_{ij} = 1$
- Perron-Frobenius: if **A** - stochastic, then $\lambda_\rho = \rho(A) = 1$
- If **A** - row stochastic, it has right eigenvector $e = (1, 1, \dots, 1)^T$,
 $Ae = \lambda e$, $\sum_j A_{ij} e_j = 1 \cdot e_i$, $\lambda = 1$
- **T**: - has largest eigenvalue $\lambda_1 = 1$
Tp = λp - corresponding right (column) eigenvector
 $p_1 = e = (1, 1, \dots, 1)^T = \text{const}$
pT = $p\lambda$, - corresponding left (row) eigenvector all positive $p_{1i} > 0$

Aperiodic graph

- A graph is *aperiodic* if the greatest common divisor of the lengths of its cycles is one
(there is no integer $k > 1$ that divides the length of every cycle of the graph)



- Convergence: strongly connected and aperiodic graph

Limiting belief

$$p(t) = Tp(t-1), t \rightarrow \infty, p(t) \rightarrow p^\infty$$

$$p^\infty = Tp^\infty, \lambda_1 = 1, p^\infty = \begin{pmatrix} c \\ \dots \\ c \end{pmatrix}$$

$$p^\infty = \lim_{t \rightarrow \infty} T^t p(0),$$

$$p^\infty = \begin{pmatrix} c \\ \dots \\ c \end{pmatrix} = \begin{pmatrix} T_{11} & \dots & T_{1n} \\ \dots & \dots & \dots \\ T_{n1} & \dots & T_{nn} \end{pmatrix}^t \begin{pmatrix} p_1(0) \\ \dots \\ p_n(0) \end{pmatrix}$$

$$\lim_{t \rightarrow \infty} \begin{pmatrix} T_{11} & \dots & T_{1n} \\ \dots & \dots & \dots \\ T_{n1} & \dots & T_{nn} \end{pmatrix}^t = \begin{pmatrix} s_1 & s_2 & \dots & s_n \\ s_1 & s_2 & \dots & s_n \\ s_1 & s_2 & \dots & s_n \end{pmatrix} = S = \begin{pmatrix} \vec{s} \\ \vec{s} \\ \vec{s} \end{pmatrix}$$

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}^{20} = \begin{pmatrix} 0.27 & 0.36 & 0.36 \\ 0.27 & 0.36 & 0.36 \\ 0.27 & 0.36 & 0.36 \end{pmatrix}$$

Social influence

- $p^\infty = \lim_{t \rightarrow \infty} T^t p(0) = Sp(0)$
 $p^\infty = \lim_{t \rightarrow \infty} T^t p(1) = \lim_{t \rightarrow \infty} T^t T p(0) = STp(0)$

$$(S - ST)p(0) = 0, \quad S = ST, \quad \begin{pmatrix} \vec{s} \\ \vec{s} \\ \vec{s} \end{pmatrix} = \begin{pmatrix} \vec{s} \\ \vec{s} \\ \vec{s} \end{pmatrix} T$$

- s - left eigenvector T

$$\vec{s} = \vec{s} \cdot T$$

- Influence

$$p^\infty = \vec{s} \cdot p(0) = \sum_i s_i p_i(0)$$

$$\vec{s} = (0.27, 0.36, 0.36)$$

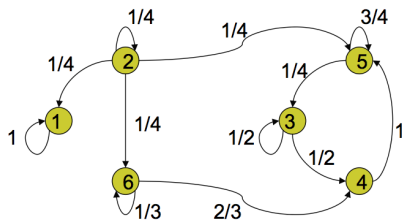
- Eigenvector centrality

$$A^T p = p$$

$$\vec{p}A = \vec{p}$$

Closed sets

- A set of nodes C is called a *closed set* if there is no direct link from the node in C to the node outside C (there is no $i \in C$ and $j \notin C$ such that $T_{ij} > 0$)



- T is convergent if and only if every set of nodes that is strongly connected and closed is aperiodic
- Every strongly connected closed and aperiodic set will reach own consensus

- Time-varying updates:

$$p(t) = \left[(1 - \lambda(t))I + \lambda(t)\hat{T} \right] p(t - 1)$$

- Time-varying weights on own beliefs:

$$p(t) = D\hat{T}p(t - 1) + (I - D)p(0)$$

- Similar beliefs:

$$T_{ij}(p(t)) = \begin{cases} \frac{1}{n_i(p(t))}, & |p_i(t) - p_j(t)| < d, n_i = \#\{k : |p_i - p_k| < d\} \\ 0, & \text{otherwise} \end{cases}$$

- Close beliefs:

$$T_{ij}(p(t)) = \frac{e^{-\gamma_{ij}|p_i(t) - p_j(t)|}}{\sum_k e^{-\gamma_{ik}|p_i(t) - p_k(t)|}}$$

- Reaching a Consensus, M. DeGroot, J. Amer. Stat. Assoc., Vol 69, N 345, pp 118-121, 1974
- A Necessary and Sufficient Condition for Reaching a Consensus Using DeGroot's Method, R. Berger, J. Amer. Stat. Assoc., Vol 76, N 374, pp 415-418, 1981
- Naive Learning in Social Networks and the Wisdom of Crowds, B. Golub and M. Jackson, Amer. Econ. J. Microeconomics. 2010