

Absorbing Chains

Leonid Zhukov

School of Applied Mathematics and Information Science
National Research University Higher School of Economics

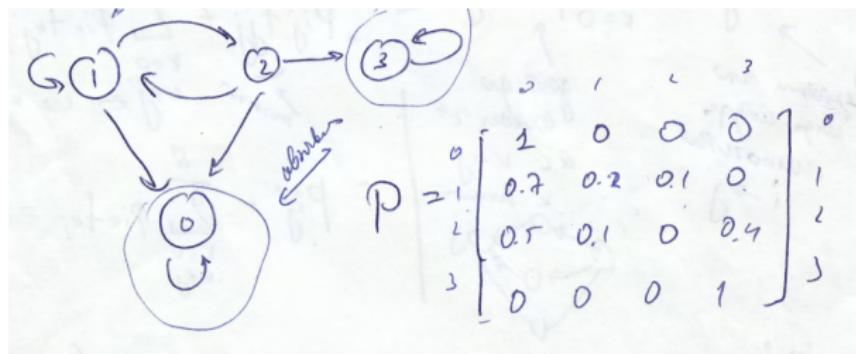
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НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ
УНИВЕРСИТЕТ

Finite Absorbing Chains

- Long run behavior, stationary state for recurrent and irreducible chains
- If there are absorbing states then:
 - how long does it take before the process gets absorbed?
 - which absorbing states are more or less likely?



First passage times

- $T_{ij} = \min\{n \geq 1 : X_n = j | X_0 = i\}$ - first passage time, the first time MC visits state j when starting from i .
- T_{ij} - random variable
- $f_{ij}^{(n)} = P\{T_{ij} = n\} = P\{X_n = j, X_{n-1} \neq j, \dots, X_1 \neq j | X_0 = i\}$, - probability that the first transition to j starting from i happens at step n .
$$f_{ij}^{(0)} = \delta_{ij}$$
- $f_{ij} = P\{T_{ij} < \infty\} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$, - probability, that the process eventually visits j
- $\mu_{ij} = E\{T_{ij}\} = \sum_{n=1}^{\infty} nP\{T_{ij} = n\} = \sum_{n=1}^{\infty} nf_{ij}^{(n)}$, - expected number of transitions for the first visit to j starting from i

Expected first passage time

- Expected time

$$\begin{aligned}\mu_{ij} &= E\{T_{ij}\} = \sum_{n=1}^{\infty} nf_{ij}^{(n)} = 1 \cdot f_{ij}^{(1)} + \sum_{n>1}^{\infty} nf_{ij}^{(n)} = \\ &= 1 \cdot p_{ij} + \sum_{k \neq j} p_{ik}(1 + \mu_{kj}) = p_{ij} + \sum_{k \neq j} p_{ik} + \sum_{k \neq j} p_{ik} \mu_{kj} = \\ &= 1 + \sum_{k \neq j} p_{ik} \mu_{kj}\end{aligned}$$

- System of equations:

$$\mu_{ij} = 1 + \sum_{k \neq j} p_{ik} \mu_{kj}, \quad i, k, j = 0..s$$

Absorbtion Probabilities

- If j - absorbing state, the first visit == absorbtion
- f_{ij} - probability that process visits j starting from i is a probability of absorption at j . i should be transitive
- Probability of absorption

$$f_{ij} = \sum_{k=0}^S p_{ik} f_{kj} = p_{ij} f_{jj} + \sum_{k \neq j} p_{ik} f_{kj} = p_{ij} + \sum_{k \neq j} p_{ik} f_{kj}$$

- System of equations:

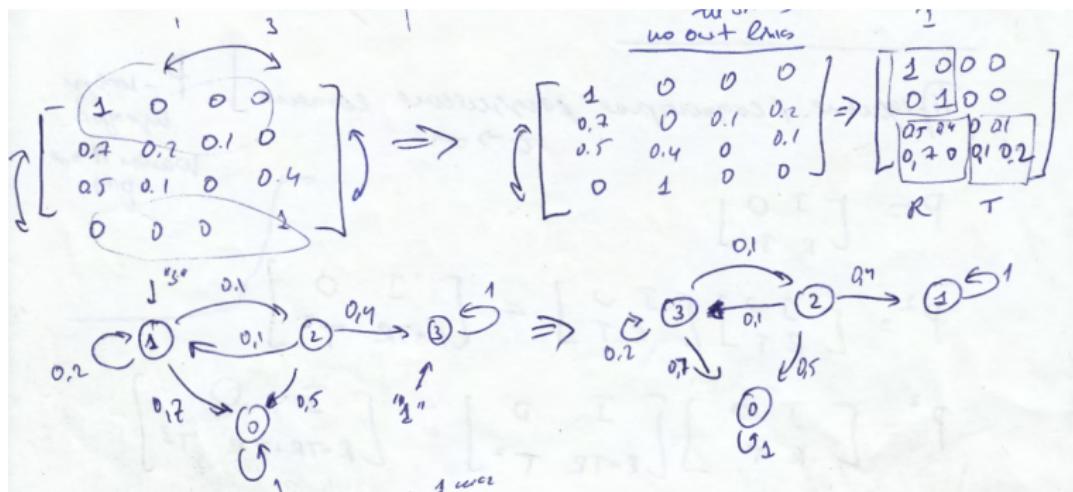
$$f_{ij} = p_{ij} + \sum_{k \in \text{trans}} p_{ik} f_{kj}$$

i - transient, j -absorbing

Transition matrix

- Transition matrix

$$P = \begin{bmatrix} I & 0 \\ R & T \end{bmatrix}$$



Transition matrix

- Transition matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{R} & \mathbf{T} \end{bmatrix}$$

$\mathbf{I}^{a \times a}$: absorbing-absorbing,

$\mathbf{R}^{t \times a}$: transient - absorbing,

$\mathbf{T}^{t \times t}$: transient - transient

- System of equations:

$$f_{ij} = p_{ij} + \sum_{k \in \text{transient}}^S p_{ik} f_{kj}$$

i - transient, j -absorbing

- Matrix form

$$\mathbf{F}^{t \times a} = \mathbf{R}^{t \times a} + \mathbf{T}^{t \times t} \cdot \mathbf{F}^{t \times a}$$

- Solution

$$\mathbf{F} = (\mathbf{I} - \mathbf{T})^{-1} \mathbf{R}$$

Limiting distribution

- n steps

$$\mathbf{P}^n = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{R} & \mathbf{T} \end{bmatrix}^n = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ (\mathbf{I} + \mathbf{T} + \dots + \mathbf{T}^n)\mathbf{R} & \mathbf{T}^n \end{bmatrix}$$

-

$$\lim_{n \rightarrow \infty} \mathbf{T}^n = \mathbf{0}$$

$$\lim_{n \rightarrow \infty} \sum_{t=0}^n \mathbf{T}^t = (\mathbf{I} - \mathbf{T})^{-1}$$

- Limiting values

$$\lim_{n \rightarrow \infty} \mathbf{P}^n = \lim_{n \rightarrow \infty} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ (\mathbf{I} - \mathbf{T})^{-1}\mathbf{R} & \mathbf{0} \end{bmatrix}^n = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{F} & \mathbf{0} \end{bmatrix}$$