Node and Link Analysis

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Centrality Measures

Sociology. Linton Freeman, 1979. Most "important" actors: actor location in the social network

- Actor centrality - involvement with other actors, many ties, source or recipient
- Actor prestige - recipient (object) of many ties, ties directed to an actor

Three graphs:
- Star graph
- Circle graph
- Line graph
Three graphs

![Graphs](image)

Leonid E. Zhukov (HSE)
Degree Centrality

Degree centrality

\[ C_D(i) = k(i) = \sum_j A_{ij} = \sum_j A_{ji} \]

Normalized degree centrality

\[ C_D^*(i) = \frac{1}{n-1} C_D(i) \]

High centrality degree - direct contact with many other actors
Low degree - not active, peripheral
Closeness Centrality

Closeness centrality
How close an actor to all the other actors in network

\[ C_C(i) = \left[ \sum_j d(i, j) \right]^{-1} \]

Normalized closeness centrality

\[ C_C^*(i) = (n - 1)C_C(i) \]

Actor in the center can quickly interact with all others, short communication path to others, minimal number of steps to reach others
Betweenness Centrality

Number of shortest paths going through the actor $\sigma_{st}(i)$

$$C_B(i) = \sum_{s \neq t \neq i} \frac{\sigma_{st}(i)}{\sigma_{st}}$$

Normalized betweenness centrality

$$C_B^*(i) = \frac{2}{(n-1)(n-2)} C_B(i)$$

Probability that a communication from $s$ to $t$ will go through $i$ (geodesics)

Edge betweenness
Degree prestige

\[ P_D(i) = k_{in}(i) = \sum_j A_{ji} \]

Normalized degree prestige

\[ P^*_D(i) = \frac{1}{n-1} P_D(i) \]

Prestigious actors receive many nominations
Proximity Prestige

Proximity prestige
Influence domain - set of actors that can reach $i$ directly and indirectly. $I_i$ - size of influence domain. Average distance $\frac{\sum_j d(j, i)}{I_i}$

$$P_p(i) = \frac{l_i/(n - 1)}{\sum_j d(j, i)/l_i}$$
Centralization (network measure) - how central the most central node in the network in relation to all other nodes.

\[
C_x = \frac{\sum_i^N [C_x(p^*_i) - C_x(p_i)]}{\max \sum_i^N [C_x(p^*_i) - C_x(p_i)]}
\]

\(C_x\) - one of the centrality measures
Graph $G(n, m)$ and adjacency matrix $A_{ij}$, edge $i \rightarrow j$
Leo Katz, 1953.
Take into account status (prestige) of directly connected actors

\[ p_i \leftarrow \sum_{j \in N(i)} p_j = \sum_j A_{ji} p_j \]

\[ p_i = \sum_j A_{ji} p_j \]

\[ p = A^T p \]

\[ (I - A^T) p = 0 \]

Nontrivial solution only if \( \text{det}(I - A^T) = 0 \). Need to constraint matrix
Leo Katz, 1953

An actor gives equal parts of its prestige to all nearest neighbours

$$p_i \leftarrow \sum_j A_{ji} \frac{p_j}{k_{out}(j)} = \sum_j \frac{A_{ji}}{k_{out}(j)} p_j$$

$$\mathbf{p} = (\mathbf{D}^{-1}\mathbf{A})^T \mathbf{p}$$

where \(D_{ii} = \max(k_i, 1)\)

\(\mathbf{D}^{-1}\mathbf{A}\) - stochastic matrix, \(\sum_j (\mathbf{D}^{-1}\mathbf{A})_{ij} = 1\), guaranteed \(\lambda_{max} = 1\)

$$(\mathbf{D}^{-1}\mathbf{A})^T \mathbf{p} = \lambda \mathbf{p}$$
Eigenvector Centrality

Phillip Bonacich, 1972.

\[ c_i \leftarrow \sum_j A_{ji} c_j \]

\[ c_i = \frac{1}{\kappa} \sum_j A_{ji} c_j \]

\[ \kappa c = A^T c \]

\[ (\kappa I - A^T)p = 0 \]

Nontrivial solution only when \( det(\kappa I - A^T) = 0 \). Eigenvalue problem. Choose eigenvector corresponding do maximum eigenvalue:

\[ \kappa_{max} = \kappa_1, \ c = c_1 \]
Parametrized centrality measure $c(\alpha, \beta)$

$$c_i = \sum_j (\alpha + \beta c_j)A_{ji}$$

$$c = \alpha A^T e + \beta A^T c$$

$$c = \alpha (I - \beta A^T)^{-1} A^T e$$

$\alpha$ - found from normalizaion $\|c\|_2 = \sum c_i^2 = 1$

$\beta$ - parameter, degree and direction of dependence on others
Graph $G(n, m)$

- zero out degree nodes, $k_{out}(i) = 0$
- zero in degree nodes, $k_{in}(i) = 0$
Real World

Bow tie structure of the web
Oscar Perron, 1907, Georg Frobenius, 1912.  
Eigenvalue problem: 

\[ Pp = \lambda p \]

Perron-Frobenius theorem: Real square matrix with positive entries 
- stochastic (non-negative and rows sum up to one) 
- irreducible (strongly connected graph) 
- aperiodic 

then unique largest eigenvalue \( \lambda_{max} = 1 \), with positive left eigenvector and power iterations converges to it. Solution satisfies \( |p|_1 = 1 \) 

Stationary distribution of Markov chain
PageRank

Sergey Brin and Larry Page, 1998

Transition matrix:

\[ \mathbf{P} = \mathbf{D}^{-1} \mathbf{A} \]

Stochastic matrix:

\[ \mathbf{P}' = \mathbf{P} + \frac{\mathbf{d} \mathbf{e}^T}{n} \]

PageRank matrix:

\[ \mathbf{P}'' = \alpha \mathbf{P}' + (1 - \alpha) \frac{\mathbf{e} \mathbf{e}^T}{n} \]

Eigenvalue problem (choose solution with \( \lambda = 1 \)):

\[ \mathbf{P}''^T \mathbf{p} = \lambda \mathbf{p} \]

Notations:

\( \mathbf{e} \) - unit column vector, \( \mathbf{d} \) - absorbing nodes indicator vector (column)
PageRank computations

- Eigenvalue problem

\[
\left[ \alpha \left( \left( D^{-1}A \right)^T + \frac{ed^T}{n} \right) + (1 - \alpha) \frac{ee^T}{n} \right] p = \lambda p
\]

- Power iterations

\[
p \leftarrow \alpha(D^{-1}A)^T p + \frac{\alpha e}{n}(d^T p) + (1 - \alpha) \frac{e}{n}(e^T p)
\]

\[
p \leftarrow \frac{p}{\|p\|}
\]

- Sparse linear system \((\lambda = 1, \|p\|_1 = 1)\)

\[
\left[ I - \alpha \left( \left( D^{-1}A \right)^T + \frac{ed^T}{n} \right) \right] p = (1 - \alpha) \frac{e}{n}
\]
Hubs and Authorities

HITS, Jon Kleinberg, 1999
Citation networks. Reviews vs original research (authoritative) papers

- authorities, contain useful information, $a_i$
- hubs, contains links to authorities, $h_i$

Mutual recursion

- Good authorities reffered by good hubs
  \[ a_i \leftarrow \sum_j A_{ji} h_j \]

- Good hubs point to good authorities
  \[ h_i \leftarrow \sum_j A_{ij} a_j \]
System of linear equations

\[ a = \alpha A^T h \]
\[ h = \beta A a \]

Symmetric eigenvalue problem

\[ (A^T A) a = \lambda a \]
\[ (A A^T) h = \lambda h \]

where eigenvalue \( \lambda = (\alpha \beta)^{-1} \)
The Medici family marriage network

<table>
<thead>
<tr>
<th>Marriage Network</th>
<th>Betweenness Centrality</th>
<th>Closeness Centrality</th>
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The Kendall tau rank distance is a metric that counts the number of pairwise disagreements between two ranking lists.

Kendall rank correlation coefficient, commonly referred to as Kendall’s tau coefficient

\[ \tau = \frac{n_c - n_d}{n(n-1)/2} \]

- \( n_c \) - number of concordant pairs,
- \( n_d \) - number of discordant pairs

\(-1 \leq \tau \leq 1\), perfect agreement \( \tau = 1 \), reversed \( \tau = -1 \)

Example

<table>
<thead>
<tr>
<th>Rank 1</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rank 2</td>
<td>C</td>
<td>D</td>
<td>A</td>
<td>B</td>
<td>E</td>
</tr>
</tbody>
</table>

\[ \tau = \frac{6 - 4}{5(5-1)/2} = 0.2 \]
References

- Centrality in Social Networks. Conceptual Clarification, Linton C. Freeman, Social Networks, 1, 215-239, 1979
References

- Authoritative Sources in a Hyperlinked Environment, Jon M. Kleinberg, Proc. 9th ACM-SIAM Symposium on Discrete Algorithms,