Network communities

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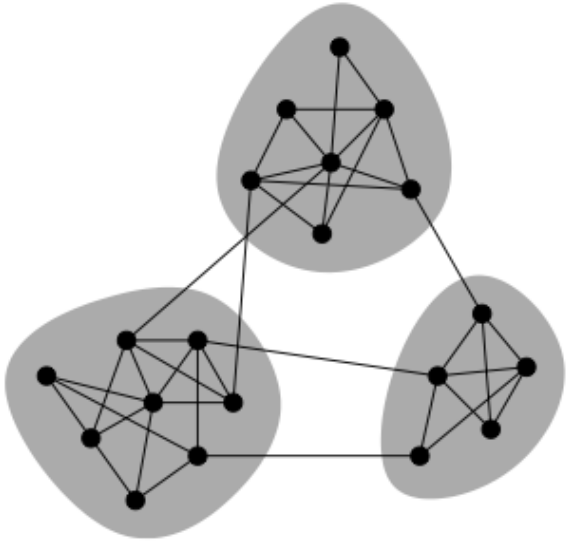
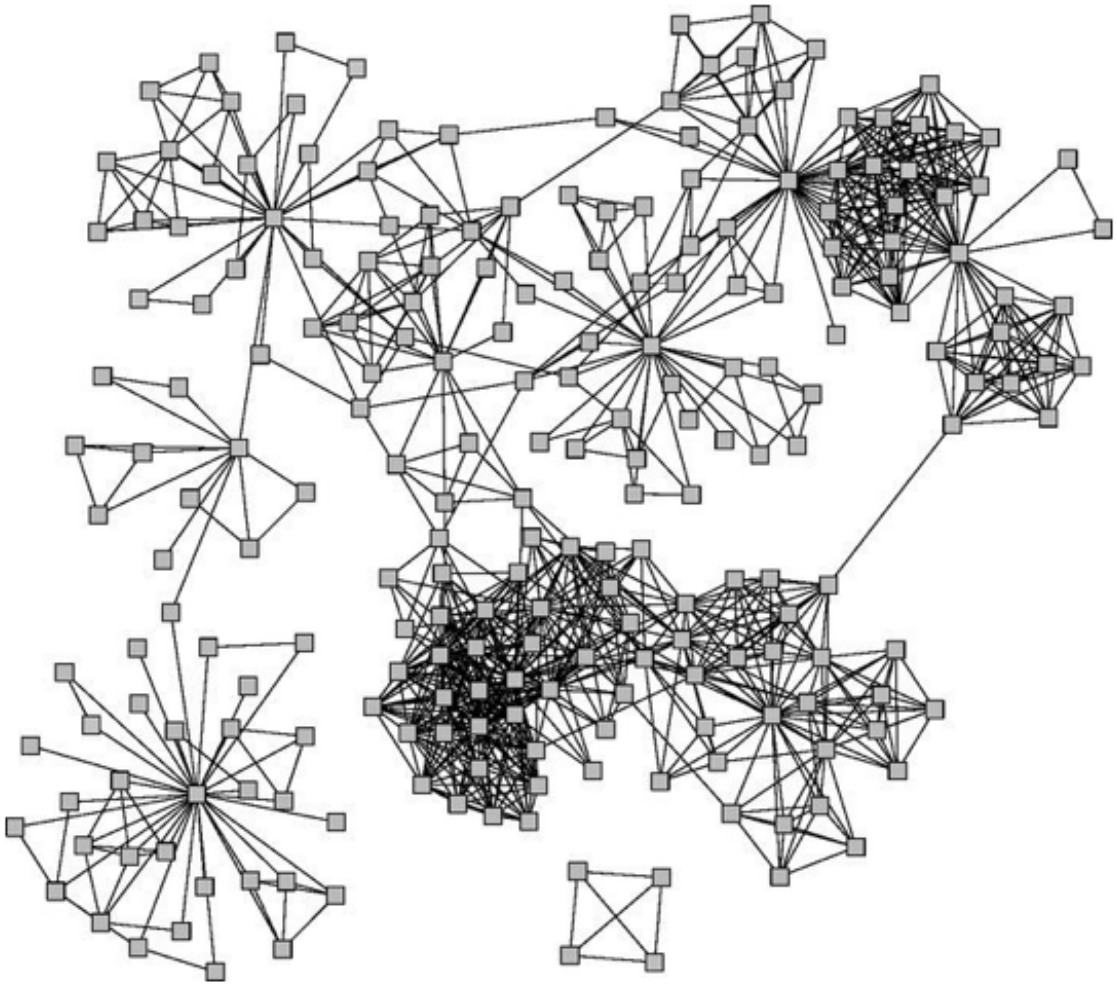
Network communities

Definition

Network community is a group of verticies such that vertices inside the group are connected with many more edges than between groups.



Network Communities



Network Communities

- graph density

$$\rho = \frac{m}{n(n-1)/2}$$

- community (cluster) density

$$\delta_{int}(C) = \frac{m_c}{n_c(n_c-1)/2}$$

- external edges density

$$\delta_{ext}(C) = \frac{m_{ext}}{n_c(n-n_c)}$$

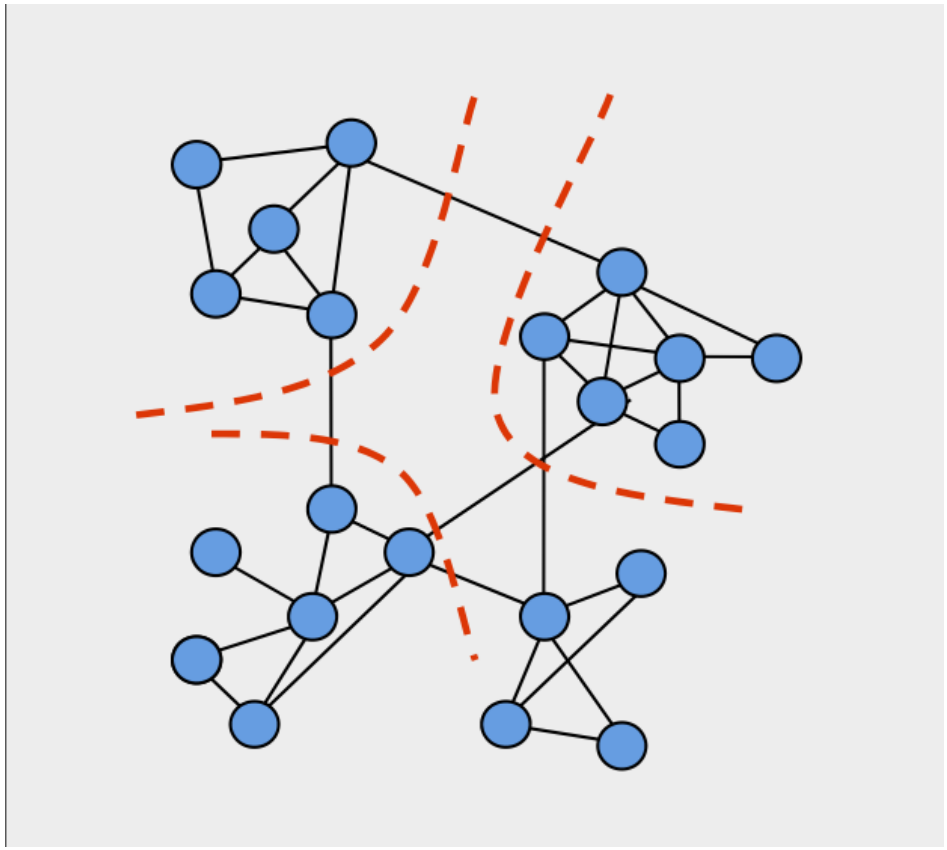
- community (cluster): $\delta_{int} > \rho$, $\delta_{ext} < \rho$
- cluster detection

$$\max(\delta_{int} - \delta_{ext})$$

Community detection

- Consider only sparse graphs $m \ll n^2$
- Each community should be connected
- Combinatorial optimization problem:
  - optimization criterion
  - optimization method
- Exact solution NP-hard
  (bi-partition: $n = n_1 + n_2$, $n!/(n_1!n_2!)$ combinations)
- Solved by greedy, approximate algorithms or heuristics
- Recursive top-down 2-way partition, multiway partiton
- Balanced class partition vs communities

Optimization criterion: graph cut



Optimization criterion : graph cut

Graph: $G(E, V)$

$||E|| = m$, $||V|| = n$, $V = V_1 + V_2$

- min cut

$$Q = cut(V_1, V_2) = \sum_{i \in V_1, j \in V_2} e_{ij}$$

- quotent cut

$$Q = \frac{cut(V_1, V_2)}{||V_1||} + \frac{cut(V_1, V_2)}{||V_2||}$$

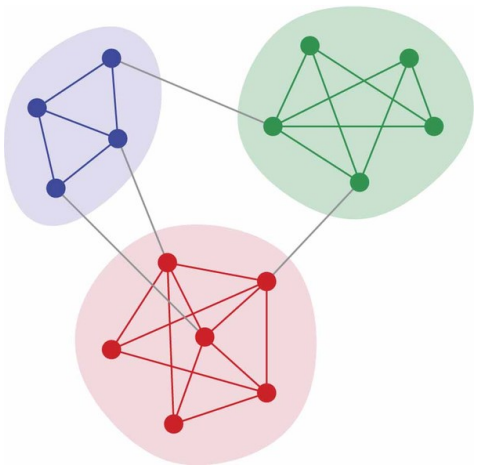
- normalized cut

$$Q = \frac{cut(V_1, V_2)}{\sum_{i \in V_1, j \in V} e_{ij}} + \frac{cut(V_1, V_2)}{\sum_{i \in V_2, j \in V} e_{ij}}$$

[Minimizaton!]

Optimization criterion: modularity

- Let n_c - number of classes, c_i - class label per node



- Modularity:

$$Q = \frac{1}{2m} \sum_{ij} \left(A_{ij} - \frac{k_i k_j}{2m} \right) \delta(c_i, c_j)$$

$$\delta(c_i, c_j) = \begin{cases} 1: & \text{if } \; c_i = c_j \\ 0: & \text{if } \; c_i \neq c_j \end{cases} \text{ - kronecker delta}$$

[Maximization!]

Modularity

- Random network
  - consider an edge e attached to node i, degree k_i
  - probability that it is attached to node j, degree k_j is $k_j/2m$
  - expected number of edges (average) between i and j is $k_i k_j/2m$
- expected number of edges within the same class, c_i- class, $\delta(c_i, c_j)$- kronecker delta

$$\langle m_c \rangle = \frac{1}{2} \sum_{ij} \frac{k_i k_j}{2m} \delta(c_i, c_j)$$

- Modularity:

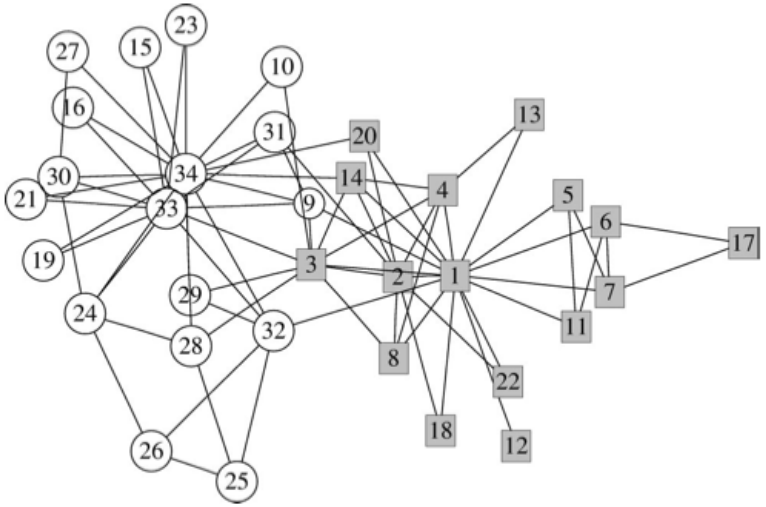
$$Q = \frac{m_c - \langle m_c \rangle}{m} = \frac{1}{2m} \sum_{ij} \left(A_{ij} - \frac{k_i k_j}{2m} \right) \delta(c_i, c_j)$$

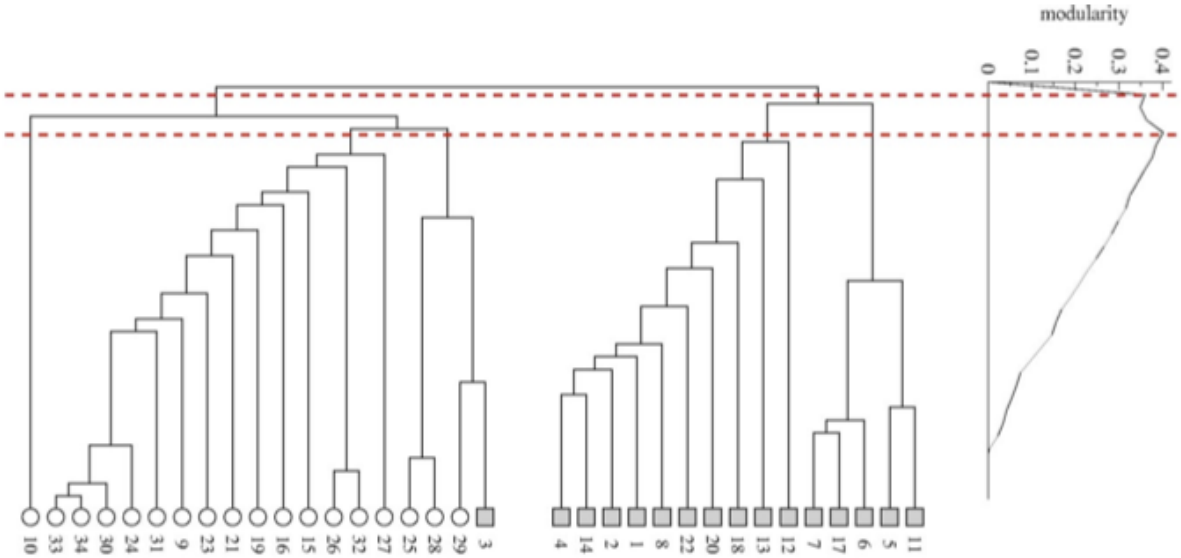
- Modularity matrix:

$$B_{ij} = A_{ij} - \frac{k_i k_j}{2m}$$

- Single class, $\delta(c_i, c_j) = 1$, $Q = \frac{1}{2m} \sum_{ij} \left(A_{ij} - \frac{k_i k_j}{2m} \right) = 0$

Dendrogram and modularity score





Optimization methods

- "Graph cut algorithms":
  - Kernighan-Lin
  - Spectral normalized cuts
  - s-t flow
  - Multilevel graph partitioning
- "Modularity algorithms":
  - Greedy
  - Spectral modularity maximizaton
- "Heuristics algorithms":
  - Edge betweenness
  - Random walks

Spectral Modularity Maximization

- Direct modularity maximization (bi-partitioning), [Newman, 2006]
- For two classes C_1, C_2 indicator variable $s = \pm 1$

$$\delta(c_i, c_j) = \frac{1}{2}(s_i s_j + 1) = \begin{cases} 1: & i, j \in C_1 \text{ or } i, j \in C_2 \\ 0: & i \in C_1, j \in C_2 \text{ or } i \in C_2, j \in C_1 \end{cases}$$

- Modularity

$$Q = \frac{1}{4m} \sum_{ij} \left(A_{ij} - \frac{k_i k_j}{2m} \right) (s_i s_j + 1) = \frac{1}{4m} \sum_{i,j} B_{ij} s_i s_j$$

where

$$B_{ij} = A_{ij} - \frac{k_i k_j}{2m}$$

Spectral Modularty Maximization

- Qudratic form:

$$Q(\mathbf{s}) = \frac{1}{4m}\mathbf{s}^T\mathbf{B}\mathbf{s}$$

- Integer optimization - NP, relaxation $s \to x$, $x \in R$
- Keep norm $||x||^2 = \sum_i x_i^2 = \mathbf{x}^T\mathbf{x} = n$
- Quadratic optimization

$$Q'(\mathbf{x}) = \frac{1}{4m}\mathbf{x}^T\mathbf{B}\mathbf{x} - \lambda(\mathbf{x}^T\mathbf{x} - n)$$

- Eigenvector problem

$$\mathbf{B}\mathbf{x}_i = \lambda_i'\mathbf{x}_i$$

- Approximate modularity

$$Q'(\mathbf{x_i}) = \frac{n}{4m}\lambda_i$$

- Maximization - maximal λ

Spectral Modularty Maximization

- Can't choose $\mathbf{s} = \mathbf{x}$k, can select optimal $\mathbf{s}$
- Decompose in the basis: $\mathbf{s} = \sum_j a_j \mathbf{x_j}$, where $a_j = \mathbf{x}_j^T \mathbf{s}$
- Modularity

$$Q(\mathbf{s}) = \frac{1}{4m}\mathbf{s}^T\mathbf{B}\mathbf{s} = \frac{1}{4m}\sum_i (\mathbf{x}_i^T\mathbf{s})^2\lambda_i$$

- max $Q(\mathbf{s})$ reached when $\lambda_1 = \lambda_{max}$ and max $\mathbf{x}_1^T\mathbf{s} = \sum_j x_{1j}s_j$
- Choose $\mathbf{s} \parallel \mathbf{x}_1$, $\mathbf{s} = sign(\mathbf{x}_1)$

Modularity maximization

Algorithm: Spectral moduldarity maximization: two-way parition

Input: adjacency matrix $\mathbf{A}$

Output: class indicator vector $\mathbf{s}$

compute $\mathbf{k} = deg(\mathbf{A})$;

compute $\mathbf{B} = \mathbf{A} - \frac{1}{2m}\mathbf{k}\mathbf{k}^T$;

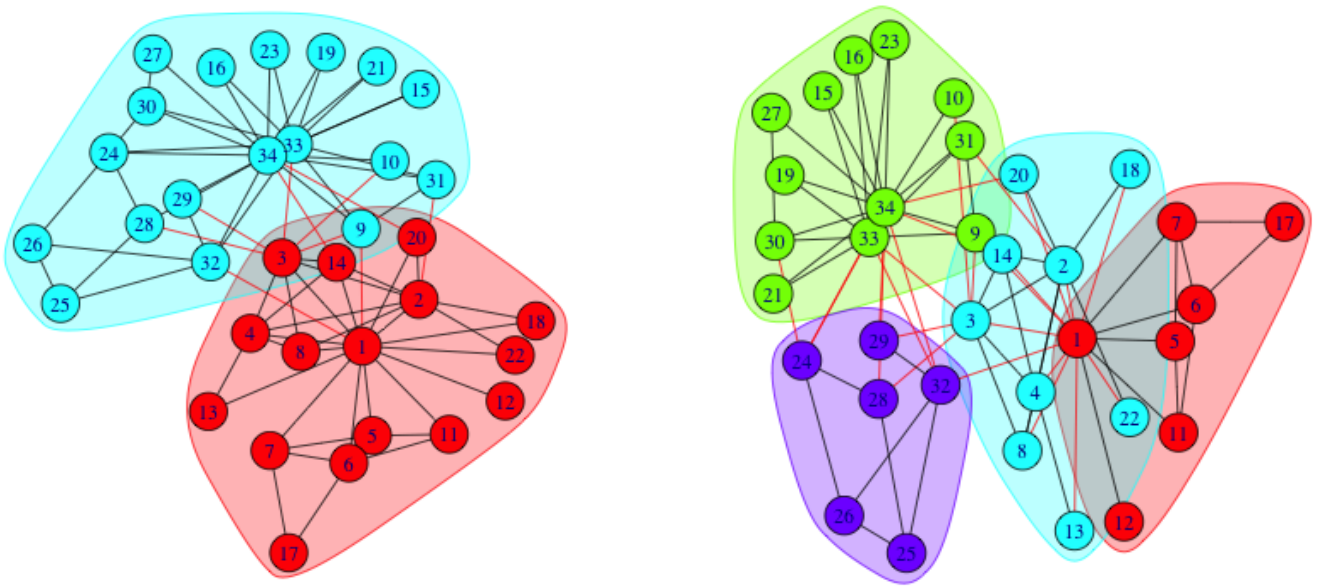
solve for maximal eigenvector $\mathbf{B}\mathbf{x} = \lambda\mathbf{x}$;

set $\mathbf{s} = sign(\mathbf{x}_1)$

Recursive bisection

Spectral modularity maximization

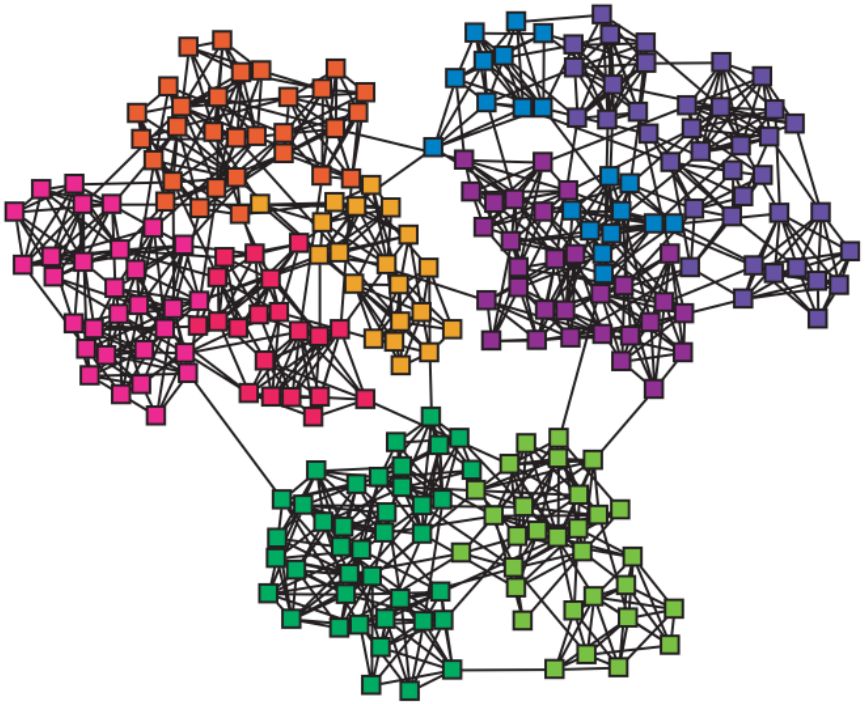
- Compute modularity matrix $B_{ij} = A_{ij} - \frac{k_i k_j}{2m}$
- Solve for maximal eigenvalue $\mathbf{Bx} = \lambda \mathbf{x}$
- set $\mathbf{s} = sign(\mathbf{x}_1)$
- recurse on each partition



Edge betweenness

Edge betweenness -number of shortest paths $\sigma_{st}(e)$ going through edge e

$$C_B(e) = \sum_{s \neq t} \frac{\sigma_{st}(e)}{\sigma_{st}}$$



Edge betweenness

Newman-Girvan, 2004

Algorithm: Edge Betweenness

Input: graph G(V,E)

Output: Dendrogram

repeat

For all $e \in E$ compute edge betweenness $C_B(e)$;

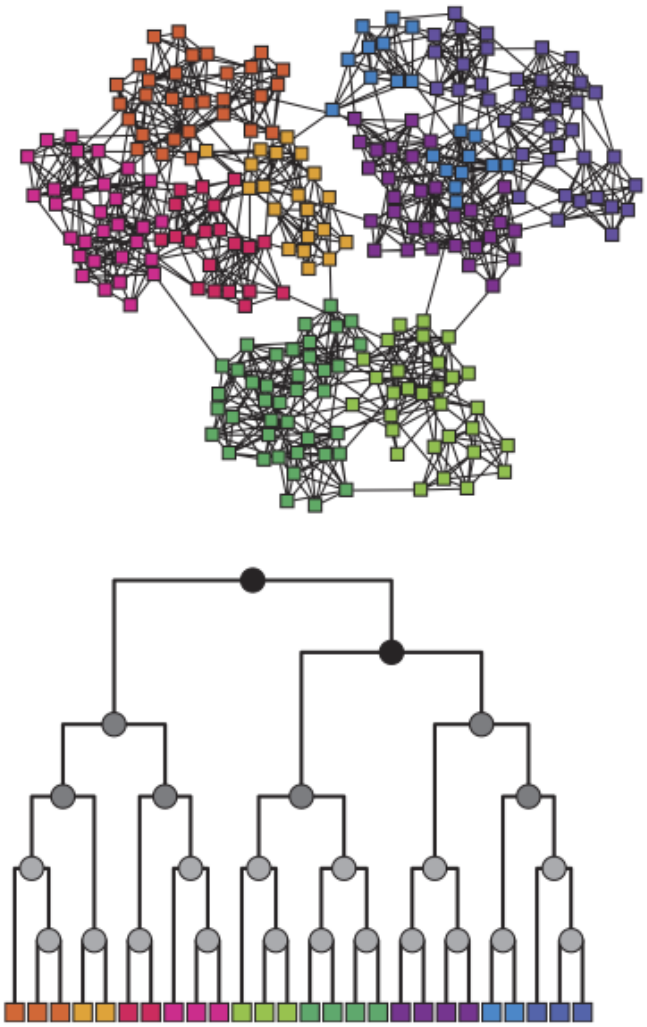
remove edge e_i with largest $C_B(e_i)$;

until *edges left*;

If bi-partition, then stop when graph splits in two components (check for connectedness)

Edge betweenness

Dendrogram



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- Modularity and community strcuture in networks, M.E.J. Newman, PNAS, vol 103, no 26, pp 8577-8582, 2006