Graph Partitioning Algorithms

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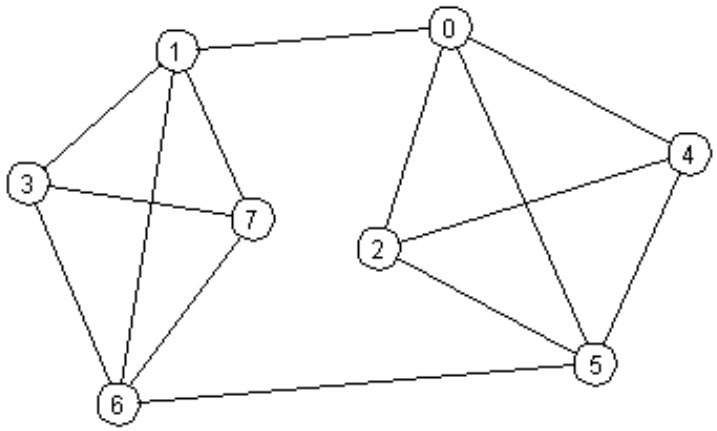
Minimum Cut

- Graph: $G(E, V)$, $|E| = m$, $|V| = n$,
- *Graph cut* - partition vertices in two disjoint subsets: $V = V_1 + V_2$
- *Cut-set* of the cut is a set of edges with endpoints in defferent partitions.
- Cut size - the number of edges in cut-set

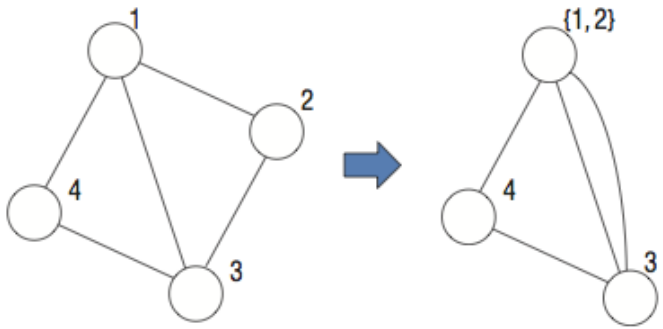
$$cut(V_1, V_2) = \sum_{i \in V_1, j \in V_2} e_{ij}$$

- *Min cut* is the smallest possible cut in the graph: $\min cut(V_1, V_2)$



Randomized Min Cut

Vertex contraction: replace conected $(v_1, v_2) \to v$, edges $E_v = E_{v_1} \cup E_{v_2}$



Randomized Min Cut

David Karger, 1993

Algorithm: Randomized Min-Cut

Input: Graph $G(V, E)$

Output: Graph min cut-set

repeat

  pick a random edge e in G;

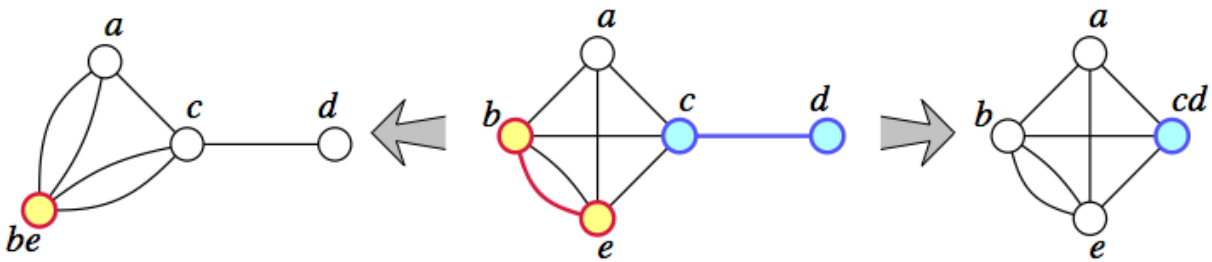
  contract its endpoints, $G' \leftarrow G \setminus e$;

until *two vertices remain*;

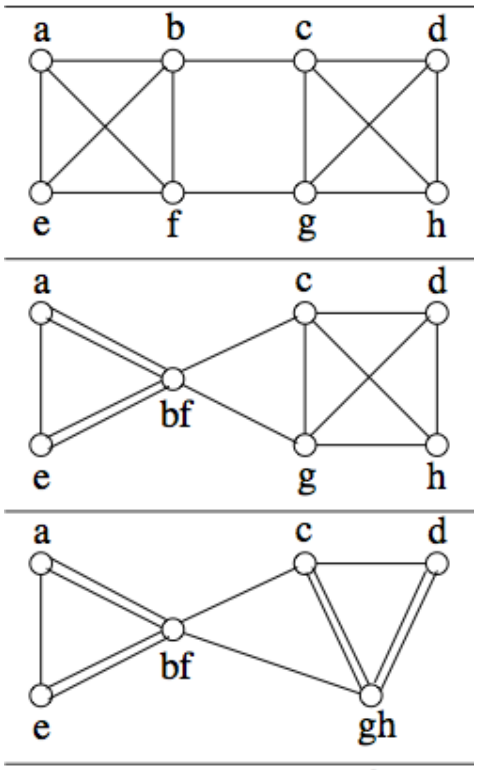
Randomized Min Cut

- Vertex contraction: $G' \leftarrow G \setminus e$
- For any cut in G' there is a cut in G with the same size (converse not true)
- Collapsing an edge can not decrease minimum cut size,

$$\min cut(G') \geq \min cut(G)$$



Randomized Min Cut



Randomized Min Cut

Graph $G(m, n)$, m-edges, n-nodes
Let $\min(cut) = k$, then every node has degree $k_i \geq k$
and there $m = \frac{1}{2}\sum_i k_i \geq \frac{nk}{2}$ edges in the graph
Probability randomly select an edge from the min cut $P(e \in \min cut) = \frac{k}{m}$
Let E_i event that on i-th step selected edge is *not* in min cut.

$$\begin{aligned} P(E_1) &\geq 1 - \frac{k}{m} \geq 1 - \frac{2}{n} = \frac{n-2}{n} \\ P(E_2|E_1) &\geq 1 - \frac{2}{n-1} = \frac{n-3}{n-1} \\ P(E_i|E_1 \cup E_2 \cup ...E_{i-1}) &\geq 1 - \frac{2}{n-i+1} = \frac{n-i-1}{n-i+1} \end{aligned}$$

$$P(E_1 \cup ..E_{n-2}) = P(E_1)P(E_2|E_1)....P(E_{n-2}|E_1 \cup E_2 \cup ...E_{n-3}) \geq$$

$$\geq \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \frac{n-5}{n-3} \cdots \frac{2}{4} = \prod_{i-1}^{n-2} \frac{n-i-1}{n-i+1} = \frac{2}{n(n-1)}$$

Randomized Min Cut

- Probability of success - all selected edges are not in min cut (what's left is min cut)

$$P(success) \geq \frac{2}{n(n-1)}$$

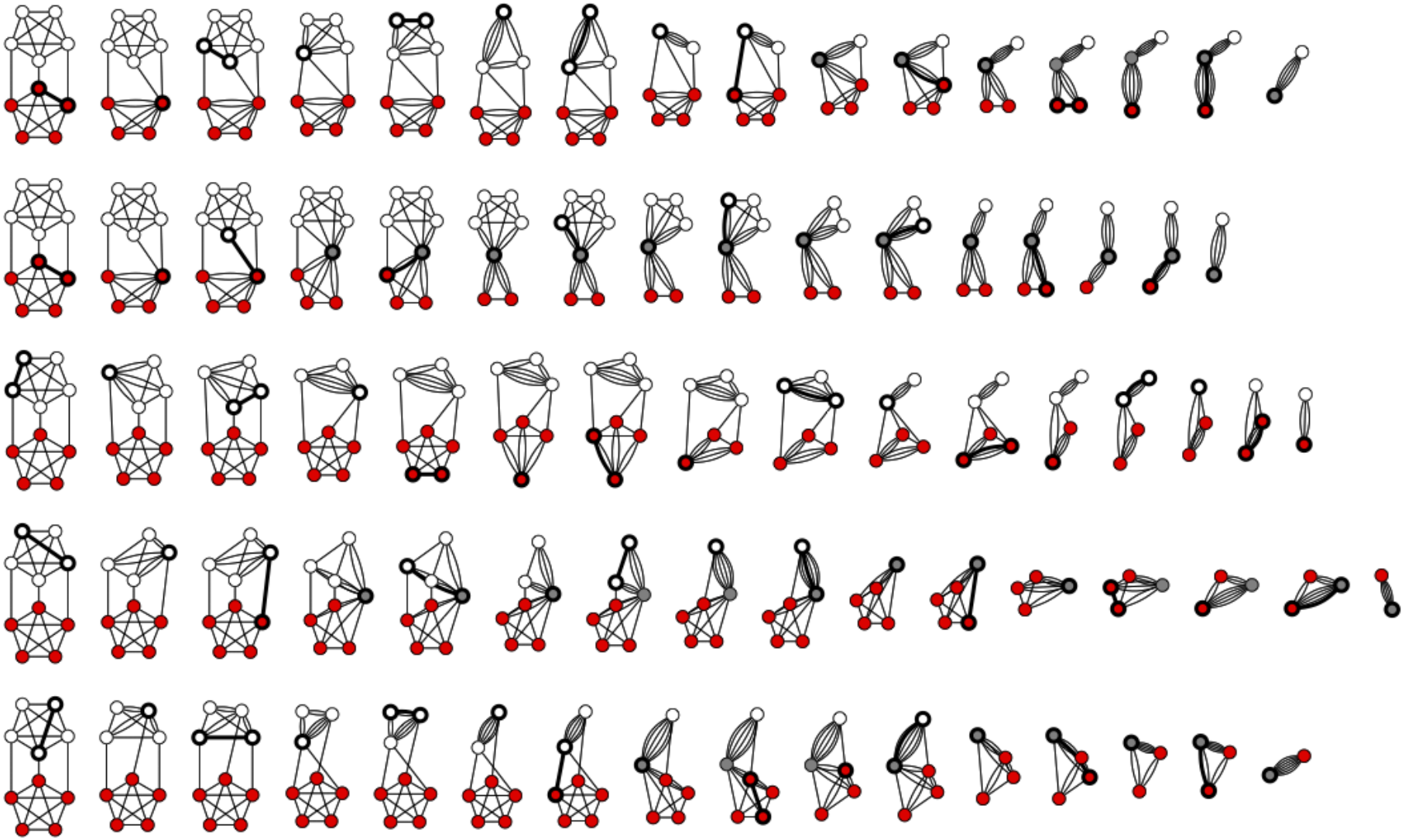
- Probability of *not* finding min cut after $N \sim n^2/2$ independent runs:

$$P(error) \leq \left(1 - \frac{2}{n(n-1)}\right)^{n^2/2} \sim \frac{1}{e} = 0.37$$

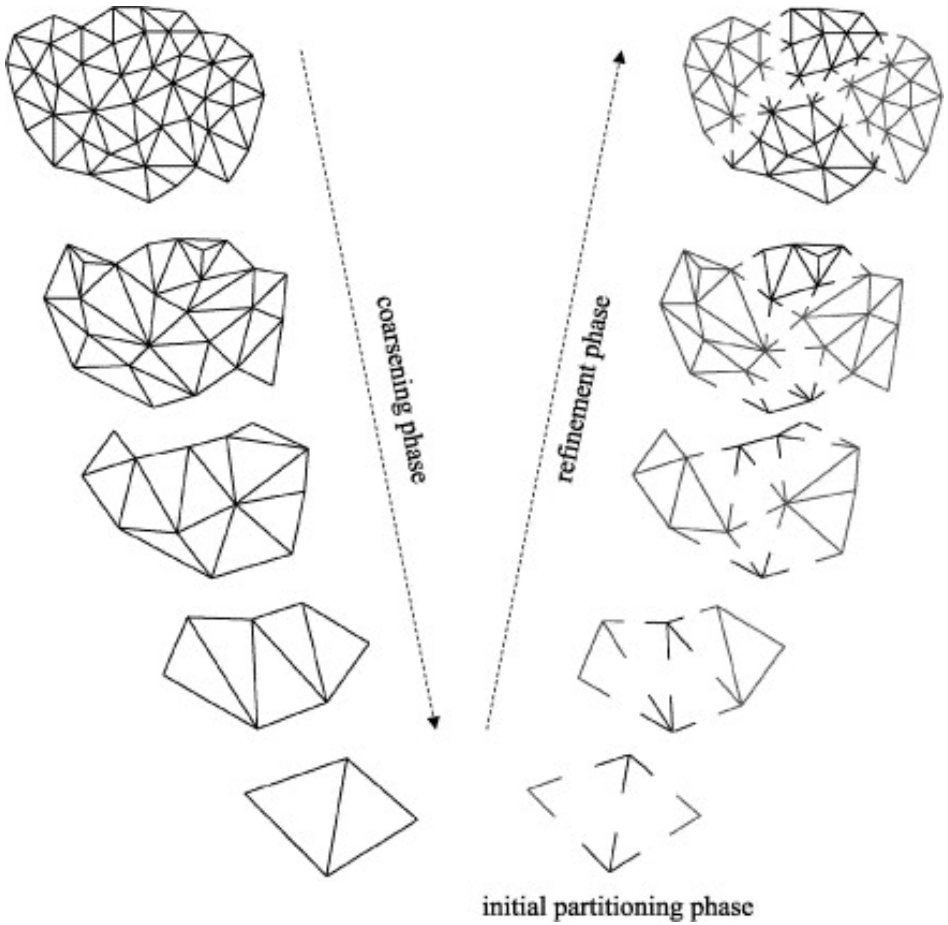
- With $N = c\frac{n(n-1)}{2} \log n$ independent runs

$$P(error) \leq \frac{1}{n^c}$$

Randomized Min Cut



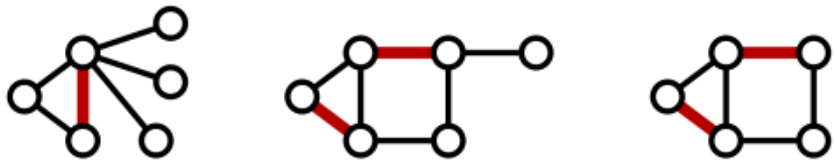
Multilevel Graph Partitioning



G. Karypis, 1998

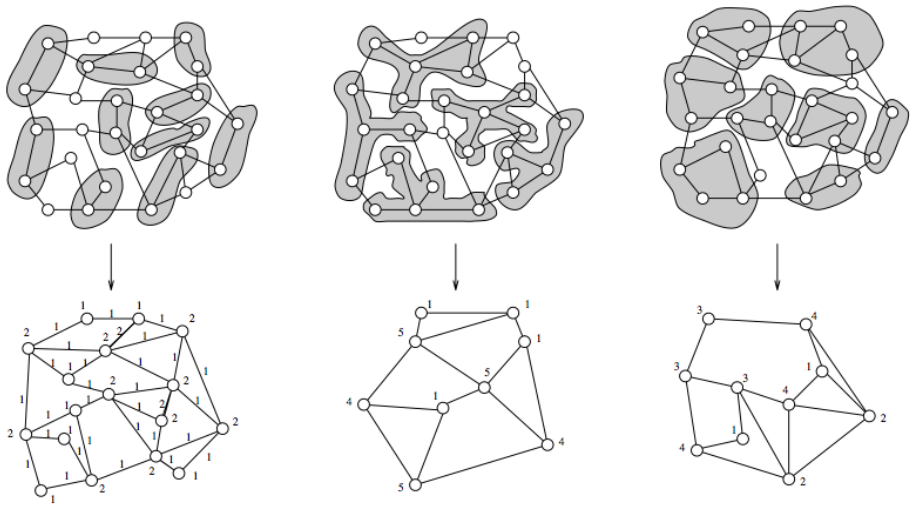
Graph Matching

- Matching - independent edge set, i.e set of edges without common veritces
- Maximal matching - if one more edge added, it is no longer a matching



Coarsening schemes

- random matching, heavy-edge matching, light-edge matching
- multinode/hypergraph
- heavy clique matching (max density)



Multilevel Graph Partitioning

George Karypis, 1998

Algorithm: Multilevel graph partitioning

Input: Graph $G(V, E)$

Output: Graph partition

1. Coarsening: $G_0 \to G_1 \to G_2 ... \to G_m$, such that $|V_0| > |V_1| > |V_2| > .. > |V_m|$
2. Partition: P_m
3. Uncoarsening: $P_m \to P_{m-1} \to P_{m-2} ... \to P_0$

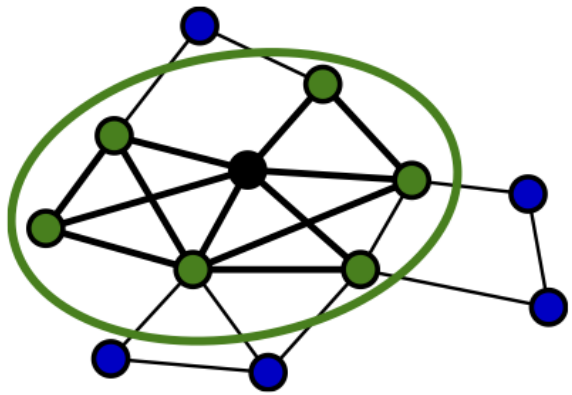
- coarsening is done by randomized maximal matching
- partition on coarse graph can be done by greedy or advanced algorithms

Local clustering algorithm

- Conductance of the vertex set S

$$\phi(S) = \frac{cut(S, V \backslash S)}{\min(vol(S), vol(S \backslash V))}$$

  where $vol(S) = \sum_{i \in S} k_i$ - sum of all node degrees in the set
- Conductance is a probability of picking up an edge from the smaller set that crosses the cut. It is also probability that one-step random walk starting in the cluster will leave the cluster
- Example: $cut(S) = 7$, $vol(S) = 33$, $vol(V \backslash S) = 11$, $\phi(S) = 7/11$



Local clustering algorithm

- Cheeger inequality

$$(1/2)\lambda_2 \leq \min_{S \in V} \phi(S) \leq \sqrt{2\lambda_2}$$

- λ_2 - second smallest eigenvalue of normalizes graph Laplacian

$$L = D^{-1/2}(D - A)D^{-1/2}, \; D = diag(d(i))$$

Local clustering algorihtm

Spielman, 2003/2008

Algorithm: Nibble

Input: Graph $G, q_0(v_0), \phi_0$

Output: Graph partition S

for $t = 1 : t_{last}$ **do**

$q_t = Mr_{t-1}$;

$r_t(i) = q_t(i) \;\; if \;\; q_t(i)/d(i) > \epsilon, else \; 0$;

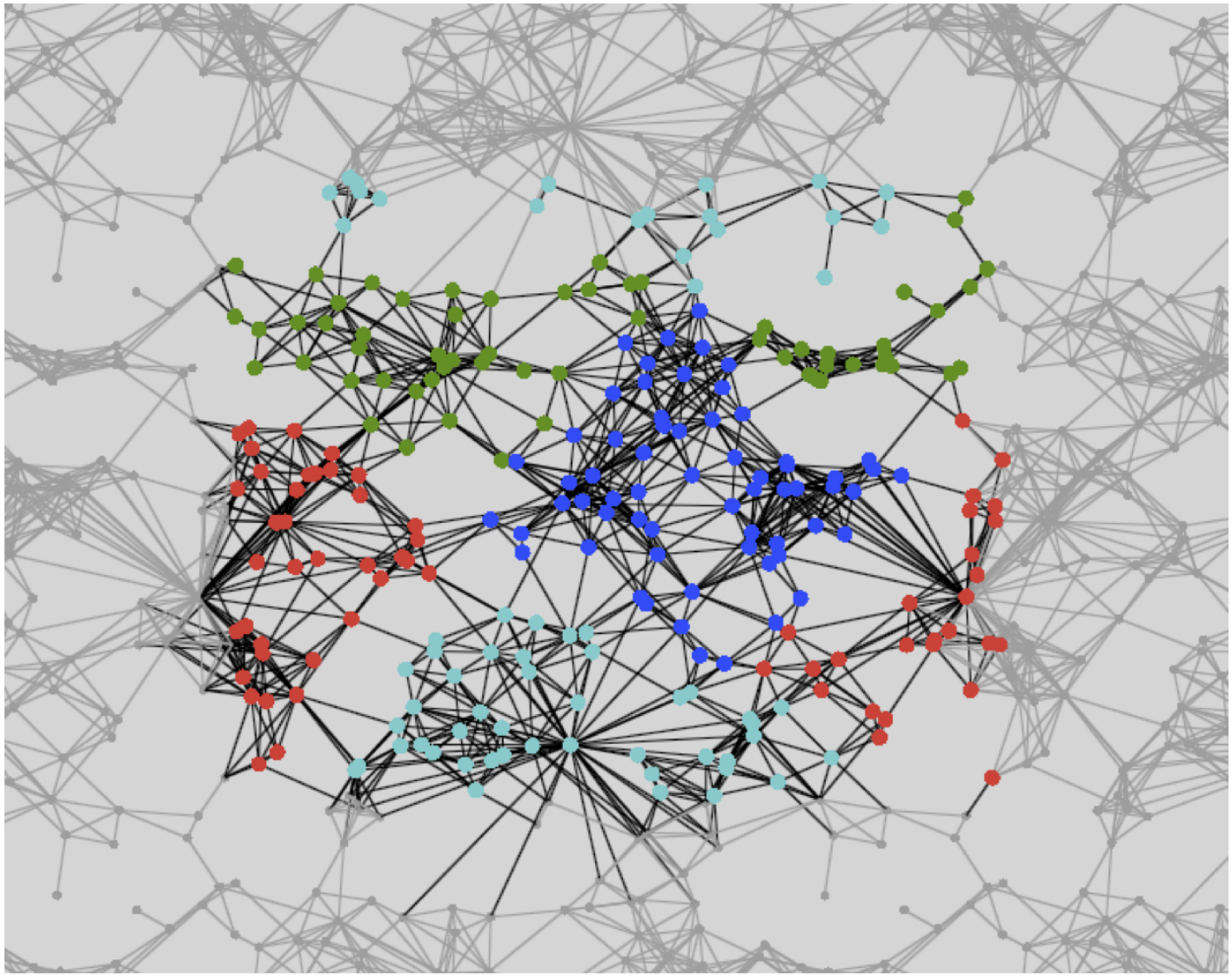
sort i descending based on $r_{t_{last}}(i)/d(i)$;

sweep from top $\phi(S\{i = 1..j\}) < \phi_0$ or $\phi(S\{i = j + 1..n\}) < \phi_0$;

- Random walk:

$$M = (AD^{-1} + I)/2, \quad D = diag(d(i))$$

Multiple clusters



D. Gleich, 2013

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