Epidemics on networks

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Epidemic models on networks

- network of potential contacts (adjacency matrix $A$)
- probabilistic model (state of a node):
  $s_i(t)$ - probability that at $t$ node $i$ is susceptible
  $x_i(t)$ - probability that at $t$ node $i$ is infected
  $r_i(t)$ - probability that at $t$ node $i$ is recovered
- from deterministic to probabilistic description
- connected component - all nodes reachable
- network is undirected (matrix $A$ is symmetric)
SI model

- SI Model
  
  $$S \rightarrow I$$

- Probabilites that node $i$: $s_i(t)$ - susceptible, $x_i(t)$ -infected at $t$
  
  $$x_i(t) + s_i(t) = 1$$

- $\beta$ - infection rate, probability to get infected in a unit time
  
  $$x_i(t + \delta t) = x_i(t) + \beta s_i \sum_j A_{ij} x_j \delta t$$

- Infection equation
  
  $$\frac{dx_i(t)}{dt} = \beta s_i(t) \sum_j A_{ij} x_j(t)$$
  
  $$\frac{ds_i(t)}{dt} = -\beta s_i(t) \sum_j A_{ij} x_j(t)$$
SI model

- Differential equation

\[
\frac{dx_i(t)}{dt} = \beta (1 - x_i(t)) \sum_j A_{ij} x_j
\]

- Early time approximation, \( t \to 0, \quad x_i(t) \ll 1 \)

\[
\frac{dx_i(t)}{dt} = \beta \sum_j A_{ij} x_j
\]

\[
\frac{dx(t)}{dt} = \beta A x(t)
\]

- Solution in the basis

\[
A v_k = \lambda_k v_k
\]

\[
x(t) = \sum_k a_k(t) v_k
\]
SI model

\[ \sum_k \frac{da_k}{dt} \mathbf{v}_k = \beta \sum_k \mathbf{A} a_k(t) \mathbf{v}_k = \beta \sum_k a_k(t) \lambda_k \mathbf{v}_k \]

\[ \frac{da_k(t)}{dt} = \beta \lambda_k a_k(t) \]

\[ a_k(t) = a_k(0) e^{\beta \lambda_k t}, \quad a_k(0) = \mathbf{v}_k^T \mathbf{x}(0) \]

- **Solution**

\[ \mathbf{x}(t) = \sum_k a_k(0) e^{\lambda_k \beta t} \mathbf{v}_k \]

- \( t \to 0, \lambda_{max} = \lambda_1 > \lambda_k \)

\[ \mathbf{x}(t) = \mathbf{v}_1 e^{\lambda_1 \beta t} \]

1. growth rate of infections depends on \( \lambda_1 \)
2. probability of infection of nodes depends on \( \mathbf{v}_1 \), i.e \( \mathbf{v}_{1i} \)
SI model

- late-time approximation, $t \to \infty$, $x_i(t) \to const$

\[
\frac{dx_i(t)}{dt} = \beta (1 - x_i(t)) \sum_j A_{ij} x_j = 0
\]

$Ax \neq 0$ since $\lambda_{min} \neq 0$, $1 - x_i(t) \approx 0$

- All nodes in connected component get infected $t \to \infty$, $x_i(t) \to 1$

- Connected component structure and distribution. Does initially infected node belong to GCC?
SI model
SIS Model

$S \rightarrow I \rightarrow S$

Probabilities that node $i$: $s_i(t)$ - susceptible, $x_i(t)$ - infected at $t$

$x_i(t) + s_i(t) = 1$

$\beta$ - infection rate, $\gamma$ - recovery rate

\[
\frac{dx_i(t)}{dt} = \beta s_i(t) \sum_j A_{ij} x_j(t) - \gamma x_i
\]

\[
\frac{ds_i(t)}{dt} = -\beta s_i(t) \sum_j A_{ij} x_j(t) + \gamma x_i
\]
SIS model

- Differential equation

\[ \frac{dx_i(t)}{dt} = \beta (1 - x_i(t)) \sum_j A_{ij} x_j - \gamma x_i \]

- early time approximation, \( x_i(t) \ll 1 \)

\[ \frac{dx_i(t)}{dt} = \beta \sum_j A_{ij} x_j - \gamma x_i \]

\[ \frac{dx_i(t)}{dt} = \beta \sum_j (A_{ij} - \frac{\gamma}{\beta} \delta_{ij}) x_j \]

\[ \frac{d\mathbf{x}(t)}{dt} = \beta (\mathbf{A} - (\frac{\gamma}{\beta}) \mathbf{I}) \mathbf{x}(t) \]

\[ \frac{d\mathbf{x}(t)}{dt} = \beta \mathbf{M} \mathbf{x}(t), \quad \mathbf{M} = \mathbf{A} - (\frac{\gamma}{\beta}) \mathbf{I} \]
SIS model

- Eigenvector basis

\[
Mv_k' = \lambda_k'v_k', \quad M = A - \left(\frac{\gamma}{\beta}\right)I, \quad Av_k = \lambda_k v_k
\]

\[
v_k' = v_k, \quad \lambda_k' = \lambda_k - \frac{\gamma}{\beta}
\]

- Solution

\[
x(t) = \sum_k a_k(t)v_k' = \sum_k a_k(0)v_k' e^{\lambda_k'\beta t} = \sum_k a_k(0)v_k e^{(\beta\lambda_k - \gamma)t}
\]

- \(\lambda_1 \geq \lambda_k\), critical: \(\beta\lambda_1 = \gamma\)
  - if \(\beta\lambda_1 > \gamma\), \(x(t) \to v_1 e^{(\beta\lambda_1 - \gamma)t}\) - growth
  - if \(\beta\lambda_1 < \gamma\), \(x(t) \to 0\) - decay
Epidemic threshold $R_0$:
- if $\frac{\beta}{\gamma} < R_0$ - infection dies over time
- if $\frac{\beta}{\gamma} > R_0$ - infection survives and becomes epidemic

In SIS model:

$$R_0 = \frac{1}{\lambda_1}, \quad \mathbf{A} \mathbf{v}_1 = \lambda_1 \mathbf{v}_1$$
SIS model

long time $t \to \infty$:

- $x_i(t) \to \text{const}$

\[
\frac{dx_i(t)}{dt} = \beta(1 - x_i) \sum_j A_{ij} x_j - \gamma x_i = 0
\]

\[
x_i = \frac{\sum_j A_{ij} x_j}{\gamma + \sum_j A_{ij} x_j}
\]

Above the epidemic threshold ($\beta/\gamma > R_0$)

- if $\beta \gg \gamma$, $x_i(t) \to 1$
- if $\beta \sim \gamma$, $x_i \frac{\gamma}{\beta} = \sum_j A_{ij} x_j$, then $\lambda_1 = \frac{\gamma}{\beta}$, $x_i(t) \to (v_1)_i$
SIR model

- SIR Model

\[ S \rightarrow I \rightarrow R \]

- Probabilities \( s_i(t) \) - susceptible, \( x_i(t) \) - infected, \( r_i(t) \) - recovered

\[ s_i(t) + x_i(t) + r_i(t) = 1 \]

- \( \beta \) - infection rate, \( \gamma \) - recovery rate

- Infection equation:

\[
\begin{align*}
\frac{ds_i}{dt} &= -\beta s_i \sum_j A_{ij} x_j \\
\frac{dx_i}{dt} &= \beta s_i \sum_j A_{ij} x_j - \gamma x_i \\
\frac{dr_i}{dt} &= \gamma x_i
\end{align*}
\]
Differential equation

\[
\frac{dx_i(t)}{dt} = \beta(1 - r_i - x_i) \sum_j A_{ij}x_j - \gamma x_i
\]

early time, \( t \to 0 \), \( r_i \sim 0 \), SIS = SIR

\[
\frac{dx_i(t)}{dt} = \beta(1 - x_i) \sum_j A_{ij}x_j - \gamma x_i
\]

Solution

\[
x(t) \sim v_1 e^{(\beta \lambda_1 - \gamma)t}
\]
SIR model
Modeling SIS

1. Every node at any time step is in one state \( \{S, I\} \)
2. Initialize \( c \) nodes in state \( I \)
3. Each node stay infected \( \tau_\gamma = 1/\gamma \) time steps
4. On each time step each \( I \) node has a probability \( \beta \) to infect its nearest neighbours (NN), \( S \rightarrow I \)
5. After \( \tau_\gamma \) time steps node recovers, \( I \rightarrow S \)
Modeling SIR

1. Every node at any time step is in one state \( \{S, I, R\} \)
2. Initialize \( c \) nodes in state \( I \)
3. Each node stay infected \( \tau_\gamma = 1/\gamma \) time steps
4. On each time step each \( I \) node has a probability \( \beta \) to infect its nearest neighbours (NN), \( S \rightarrow I \)
5. After \( \tau_\gamma \) time steps node recovers, \( I \rightarrow R \)
6. Nodes \( R \) do not participate in infection propagation
Modeling SIR

(a)

(b)

(c)

(d)
Networks: random, lattice, small world, spatial, scale-free

Keeling et al, 2005
Networks: random, lattice, small world, spatial, scale-free

Keeling et al, 2005
Small-world network at different values of disorder parameter $p$

Kuperman et al, 2001
References