Epidemics

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Lecture outline

1. Epidemic models
   - SI model
   - SIS model
   - SIR model
2. Branching process
   - Galton-Watson process

Epidemic dynamics models

- Mathematical epidimiology
- W. O. Kermack and A. G. McKendrick, 1927
- Deterministic compartamental model (population classes) $\{S, I, T\}$
- $S(t)$ - succeptable, number of individuals not yet infected with the disease at time t
- $I(t)$ - infected, number of individuals who have been infected with the disease and are capable of spreading the disease.
- $R(t)$ - recoverd, number of individuals who have been infected and then recovered from the disease, can't be infected again or to transmit the infection to others.
- Fully-mixing model
- Closed population (no birth, death, migration)
- Models: SI, SIS, SIR, SIRS,..

SI model

- $S(t)$ -susceptible , $I(t)$ - infected

$$S \longrightarrow I$$

$$S(t) + I(t) = N$$

- β - infection/contact rate, number of contacts per unit time
- Infection equation:

$$I(t + \delta t) = I(t) + \beta \frac{S(t)}{N} I(t) \delta t$$

$$\frac{dI(t)}{dt} = \beta \frac{S(t)}{N} I(t)$$

SI model

- Fractions: $i(t) = I(t)/N$, $s(t) = S(t)/N$
- Equations

$$\begin{aligned}
\frac{di(t)}{dt} &= \beta s(t) i(t) \\
\frac{ds(t)}{dt} &= -\beta s(t) i(t)
\end{aligned}$$

$$s(t) + i(t) = 1$$

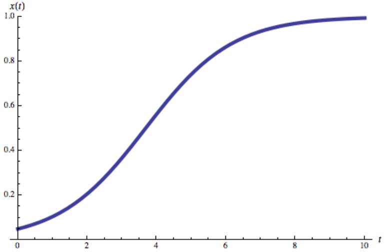
- Differential equation, $i(t = 0) = i_0$

$$\frac{di(t)}{dt} = \beta(1 - i(t)) i(t)$$

Logistic growth function

- Solution:

$$i(t) = \frac{i_0}{i_0 + (1 - i_0)e^{-\beta t}}$$



- Limit $t \to \infty$

$$i(t) \to 1$$
$$s(t) \to 0$$

in image $i_0 = 0.05, \quad \beta = 0.8$

SIS model

- $S(t)$ -susceptable , $I(t)$ - infected,

$$S \longrightarrow I \longrightarrow S$$

$$S(t) + I(t) = N$$

- β - infection rate (on contact), γ - recovery rate
- Infection equations:

$$\begin{aligned} \frac{ds}{dt} &= -\beta si + \gamma i \\ \frac{di}{dt} &= \beta si - \gamma i \end{aligned}$$

$$s + i = 1$$

- Differential equation, $i(t = 0) = i_0$

$$\frac{di}{dt} = (\beta - \gamma - i)i$$

SIS model

- Solution

$$i(t) = (1 - \frac{\gamma}{\beta})\frac{C}{C + e^{-(\beta-\gamma)t}}$$

where

$$C = \frac{\beta i_0}{\beta - \gamma - \beta i_0}$$

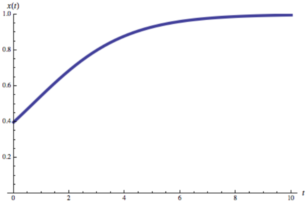
- Limit $t \to \infty$

$$\beta > \gamma \quad , \qquad i(t) \to (1 - \frac{\gamma}{\beta})$$

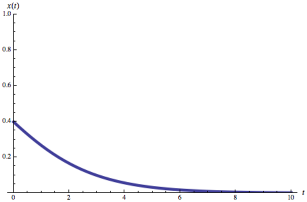
$$\beta < \gamma \quad , \qquad i(t) = i_0 e^{(\beta-\gamma)t} \to 0$$

Logistic function

- $\beta > \gamma, \quad i(t) \to (1 - \frac{\gamma}{\beta})$



- $\beta < \gamma, \quad i(t) = i_0 e^{(\beta-\gamma)t} \to 0$



SIR model

- $S(t)$ -susceptable , $I(t)$ - infected, $R(t)$ - recovered

$$S \longrightarrow I \longrightarrow R$$

$$S(t) + I(t) + R(t) = N$$

- β - infection rate, γ - recovery rate
- Infection equation:

$$\begin{aligned} \frac{ds}{dt} &= -\beta si \\ \frac{di}{dt} &= \beta si - \gamma i \\ \frac{dr}{dt} &= \gamma i \end{aligned}$$

$$s + i + r = 1$$

SIR model

- Equation

$$\frac{ds}{dt} = -\beta s \frac{dr}{dt}\frac{1}{\gamma}$$

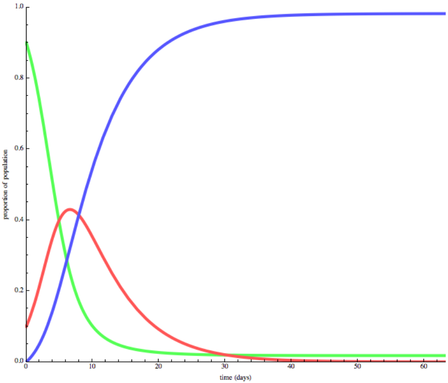
$$s = s_0 e^{-\frac{\beta}{\gamma} r}$$

$$\frac{dr}{dt} = \gamma(1 - r - s_0 e^{-\frac{\beta}{\gamma} r})$$

- Solution

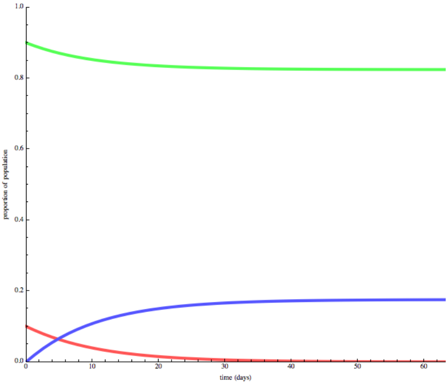
$$t = \frac{1}{\gamma}\int_0^r \frac{dr}{1 - r - s_0 e^{-\frac{\beta}{\gamma} r}}$$

SIR model



- $\frac{\beta}{\gamma} = 4$
- $i_0 = 0.1$

SIR model



- $\frac{\beta}{\gamma} = 0.5$
- $i_0 = 0.1$

SIR model

- Equation

$$\frac{dr}{dt} = \gamma(1 - r - s_0 e^{-\frac{\beta}{\gamma} r})$$

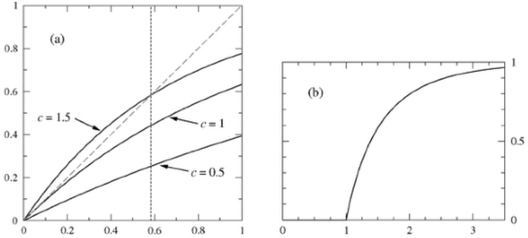
- Limits: $t \to \infty$, $\frac{dr}{dt} = 0$, $r_\infty = const$,

$$1 - r_\infty = s_0 e^{-\frac{\beta}{\gamma} r_\infty}$$

- Initial conditions: $r(0) = 0$, $i(0) = c/N$, $s(0) = 1 - c/N \approx 1$

$$1 - r_\infty = e^{-\frac{\beta}{\gamma} r_\infty}$$

SIR model



$$r_\infty = 1 - e^{-R_0 r_\infty}, \quad R_0 = \frac{\beta}{\gamma}$$

$$(r_\infty)'|_{r_\infty=0} = (1 - e^{-R_0 r_\infty})'|_{r_\infty=0},$$

$$\text{critical point: } \; R_0 = 1$$

SIR model

- r_∞ - the total size of the outbreak
- Epidemic threshold

$$\begin{array}{lllll} \text{Epidemics:} & R_0 > 1, & \beta > \gamma & , & r_\infty = const > 0 \\ \text{No epidemics:} & R_0 < 1, & \beta < \gamma & , & r_\infty \to 0 \end{array}$$

- Basic reproduction number

$$R_0 = \frac{\beta}{\gamma}$$

It is average number of people infected by a person before his recovery

$$R_0 = E[\beta\tau] = \beta \int_0^\infty \gamma\tau e^{-\gamma\tau} d\tau = \frac{\beta}{\gamma}$$

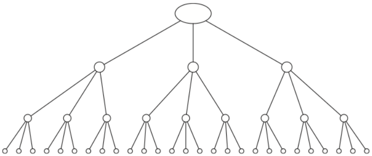
Model of contagion

Simple model of contagion (decease transmission)

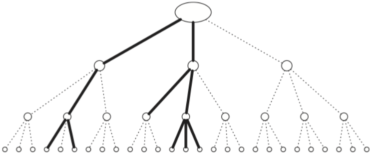
- 1st-wave: first infected person enters the population and transmits to each person he meets with probability p. Suppose he meets k people while contagious
- 2nd-wave: Each infected person from 1st wave meets k new people and independently transmits infection with probability p
- 3rd-wave: ....

This is Galton-Watson branching stochastic process (Proposed by Francis Galton 1889 as a model for extinction of family names)

Branching process



(a)



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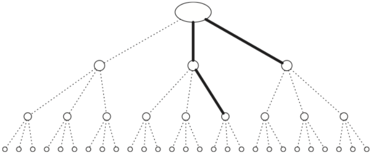


(c)

image from David Easley, Jon Kleinberg, 2010

Branching process

Random branching process:

- let ξ_i^n - number of transmitted infections by ith node on level n
- let Z_n - number of infected on level n, $Z_0 = 1$. Then:

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{(n)}$$

- If each node has k neighbors, transmits infection with probability p, Average number of infected people $E[\xi_i^n] = pk = R_0$ - basic reproductive number
- Recursion

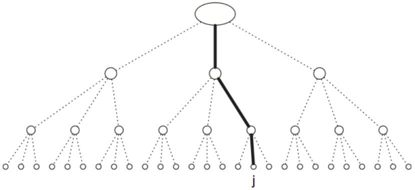
$$E[Z_{n+1}] = E[\sum_{i=1}^{Z_n} \xi_i^{(n)}] = E[\xi_i^{(n)}]\ E[Z_n] = pk\ E[Z_n]$$

$$E[Z_n] = (pk)^n = R_0^n$$

Branching process

Galton-Watson branching random process:

- if $R_0 = 1$, the mean of number of infected nodes does not change
- if $R_0 > 1$, the mean grows geometrically as R_0^n
- if $R_0 < 1$, the mean shrinks geometrically as R_0^n

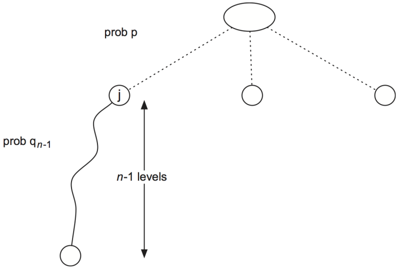


$R_0 = 1$ - point of phase transition

Branching process

Extinction probability

- let q_n - probability that infection persists n steps (levels of the tree)
- pq_{n-1} - probability that spreads through one first contact and then survives $n-1$ levels



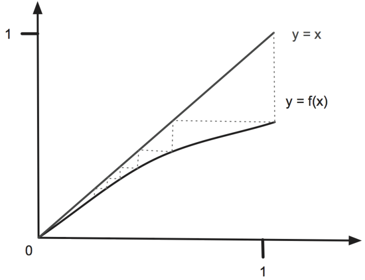
- $(1-pq_{n-1})^k$ - probability that will not spread through any of the subtries

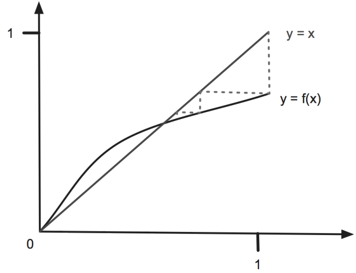
$$(1-pq_{n-1})^k = 1 - q_n$$

Branching process

- Recurrence (q_n - probability that infection persists through n steps)

$$q_n = 1 - (1 - pq_{n-1})^k$$

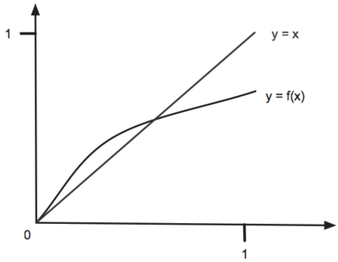




Branching process

- limiting probability $q^* = \lim_{n\to\infty} q_n$

$$q^* = 1 - (1 - pq^*)^k$$



- Slope:

$$pk(1 - pq)^{k-1}\Big|_{q=0} = 1$$

- When $R_0 = pk > 1$, there is a non zero probability of infection persists

References

- A Contribution to the Mathematical Theory of Epidemics. , Kermack, W. O. and McKendrick, A. G. , Proc. Roy. Soc. Lond. A 115, 700-721, 1927.
- The Mathematics of Infectious Disease, Herbert W. Hethcote, SIAM Review, Vol. 42, No. 4, p. 599-653, 2000