Epidemics

Leonid E. Zhukov

School of Data Analysis and Artificial Intelligence
Department of Computer Science
National Research University Higher School of Economics

Structural Analysis and Visualization of Networks
Lecture outline

1. Epidemic models
   - SI model
   - SIS model
   - SIR model

2. Branching process
   - Galton-Watson process
Epidemic dynamics models

- Mathematical epidemiology
- W. O. Kermack and A. G. McKendrick, 1927
- Deterministic compartmental model (population classes) \( \{S, I, T\} \)
- \( S(t) \) - susceptible, number of individuals not yet infected with the disease at time \( t \)
- \( I(t) \) - infected, number of individuals who have been infected with the disease and are capable of spreading the disease.
- \( R(t) \) - recovered, number of individuals who have been infected and then recovered from the disease, can’t be infected again or to transmit the infection to others.
- Fully-mixing model
- Closed population (no birth, death, migration)
- Models: SI, SIS, SIR, SIRS,...
SI model

- $S(t)$ - susceptible, $I(t)$ - infected

\[ S(t) + I(t) = N \]

- $\beta$ - infection/contact rate, number of contacts per unit time

Infection equation:

\[
I(t + \delta t) = I(t) + \beta \frac{S(t)}{N} I(t) \delta t
\]

\[
\frac{dI(t)}{dt} = \beta \frac{S(t)}{N} I(t)
\]
SI model

- Fractions: $i(t) = I(t)/N$, $s(t) = S(t)/N$
- Equations

\[
\begin{align*}
\frac{di(t)}{dt} &= \beta s(t)i(t) \\
\frac{ds(t)}{dt} &= -\beta s(t)i(t)
\end{align*}
\]

\[s(t) + i(t) = 1\]

- Differential equation, $i(t = 0) = i_0$

\[
\frac{di(t)}{dt} = \beta (1 - i(t))i(t)
\]
Logistic growth function

Solution:

\[ i(t) = \frac{i_0}{i_0 + (1 - i_0)e^{-\beta t}} \]

Limit \( t \to \infty \)

\[ i(t) \to 1 \]
\[ s(t) \to 0 \]

in image \( i_0 = 0.05, \ \beta = 0.8 \)
SIS model

- $S(t)$ - susceptible, $I(t)$ - infected,

$$S \rightarrow I \rightarrow S$$

$$S(t) + I(t) = N$$

- $\beta$ - infection rate (on contact), $\gamma$ - recovery rate

Infection equations:

$$\frac{ds}{dt} = -\beta si + \gamma i$$

$$\frac{di}{dt} = \beta si - \gamma i$$

$$s + i = 1$$

- Differential equation, $i(t = 0) = i_0$

$$\frac{di}{dt} = (\beta - \gamma - i)i$$
Solution

\[ i(t) = \left(1 - \frac{\gamma}{\beta}\right) \frac{C}{C + e^{-(\beta-\gamma)t}} \]

where

\[ C = \frac{\beta i_0}{\beta - \gamma - \beta i_0} \]

Limit \( t \to \infty \)

\[ \beta > \gamma \ , \quad i(t) \to \left(1 - \frac{\gamma}{\beta}\right) \]

\[ \beta < \gamma \ , \quad i(t) = i_0 e^{(\beta-\gamma)t} \to 0 \]
Logistic function

- $\beta > \gamma$, \hspace{1em} \( i(t) \rightarrow (1 - \frac{\gamma}{\beta}) \)

- $\beta < \gamma$, \hspace{1em} \( i(t) = i_0 e^{(\beta - \gamma)t} \rightarrow 0 \)
SIR model

- $S(t)$ - susceptible, $I(t)$ - infected, $R(t)$ - recovered

$$S \rightarrow I \rightarrow R$$

$$S(t) + I(t) + R(t) = N$$

- $\beta$ - infection rate, $\gamma$ - recovery rate

Infection equation:

$$\frac{ds}{dt} = -\beta si$$
$$\frac{di}{dt} = \beta si - \gamma i$$
$$\frac{dr}{dt} = \gamma i$$

$$s + i + r = 1$$
SIR model

- Equation

\[
\frac{ds}{dt} = -\beta s \frac{dr}{dt} \frac{1}{\gamma}
\]

\[s = s_0 e^{-\frac{\beta}{\gamma} r}\]

\[
\frac{dr}{dt} = \gamma(1 - r - s_0 e^{-\frac{\beta}{\gamma} r})
\]

- Solution

\[
t = \frac{1}{\gamma} \int_0^r \frac{dr}{1 - r - s_0 e^{-\frac{\beta}{\gamma} r}}
\]
\[ \frac{\beta}{\gamma} = 4 \]
\[ i_0 = 0.1 \]
SIR model

- $\frac{\beta}{\gamma} = 0.5$
- $i_0 = 0.1$
SIR model

- Equation
  \[
  \frac{dr}{dt} = \gamma (1 - r - s_0 e^{-\frac{\beta}{\gamma} r})
  \]

- Limits: \( t \to \infty \), \( \frac{dr}{dt} = 0 \), \( r_\infty = \text{const} \),
  \[
  1 - r_\infty = s_0 e^{-\frac{\beta}{\gamma} r_\infty}
  \]

- Initial conditions: \( r(0) = 0 \), \( i(0) = c/N \), \( s(0) = 1 - c/N \approx 1 \)
  \[
  1 - r_\infty = e^{-\frac{\beta}{\gamma} r_\infty}
  \]
SIR model

\[ r_\infty = 1 - e^{-R_0 r_\infty}, \quad R_0 = \frac{\beta}{\gamma} \]

\[ (r_\infty)'|_{r_\infty=0} = (1 - e^{-R_0 r_\infty})'|_{r_\infty=0}, \]

critical point: \( R_0 = 1 \)
SIR model

- $r_\infty$ - the total size of the outbreak
- Epidemic threshold

  Epidemics:  $R_0 > 1, \; \beta > \gamma, \; r_\infty = const > 0$

  No epidemics:  $R_0 < 1, \; \beta < \gamma, \; r_\infty \to 0$

- Basic reproduction number

  $$R_0 = \frac{\beta}{\gamma}$$

  It is average number of people infected by a person before his recovery

  $$R_0 = E[\beta \tau] = \beta \int_0^\infty \gamma \tau e^{-\gamma \tau} d\tau = \frac{\beta}{\gamma}$$
Simple model of contagion (decease transmission)

- **1st-wave**: first infected person enters the population and transmits to each person he meets with probability $p$. Suppose he meets $k$ people while contagious.
- **2nd-wave**: Each infected person from 1st wave meets $k$ new people and independently transmits infection with probability $p$.
- **3rd-wave**: ....

This is Galton-Watson branching stochastic process (Proposed by Francis Galton 1889 as a model for extinction of family names).
Branching process

image from David Easley, Jon Kleinberg, 2010
Branching process

Random branching process:

- let $\xi^n_i$ - number of transmitted infections by $i$th node on level $n$
- let $Z_n$ - number of infected on level $n$, $Z_0 = 1$. Then:

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi^{(n)}_i$$

- If each node has $k$ neighbors, transmits infection with probability $p$, Average number of infected people $E[\xi^n_i] = pk = R_0$ - basic reproductive number
- Recursion

$$E[Z_{n+1}] = E[\sum_{i=1}^{Z_n} \xi^{(n)}_i] = E[\xi^{(n)}_i] E[Z_n] = pk E[Z_n]$$

$$E[Z_n] = (pk)^n = R^n_0$$
Galton-Watson branching random process:

- if $R_0 = 1$, the mean of number of infected nodes does not change
- if $R_0 > 1$, the mean grows geometrically as $R_0^n$
- if $R_0 < 1$, the mean shrinks geometrically as $R_0^n$

$R_0 = 1$ - point of phase transition
Branching process

Extinction probability

- let $q_n$ - probability that infection persists $n$ steps (levels of the tree)
- $pq_{n-1}$ - probability that spreads through one first contact and then survives $n - 1$ levels

$(1 - pq_{n-1})^k$ - probability that will not spread through any of the subtries

$$(1 - pq_{n-1})^k = 1 - q_n$$
Branching process

- Recurrence ($q_n$ - probability that infection persists through $n$ steps)

$$q_n = 1 - (1 - pq_{n-1})^k$$
Branching process

- limiting probability \( q^* = \lim_{n \to \infty} q_n \)

\[
q^* = 1 - (1 - pq^*)^k
\]

- Slope:

\[
pk(1 - pq)^{k-1}\bigg|_{q=0} = 1
\]

- When \( R_0 = pk > 1 \), there is a non zero probability of infection persists