

# Power laws

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## Structural Analysis and Visualization of Networks



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# Continuous distribution

- Continuous random variable  $X$
- Probability density function  $p(x)$  (PDF):

$$\begin{aligned} \Pr(a \leq X \leq b) &= \int_a^b p(x)dx \\ p(x) &\geq 0 \\ \int_{-\infty}^{\infty} p(x)dx &= 1 \end{aligned}$$

- Cumulative distribution function (CDF)

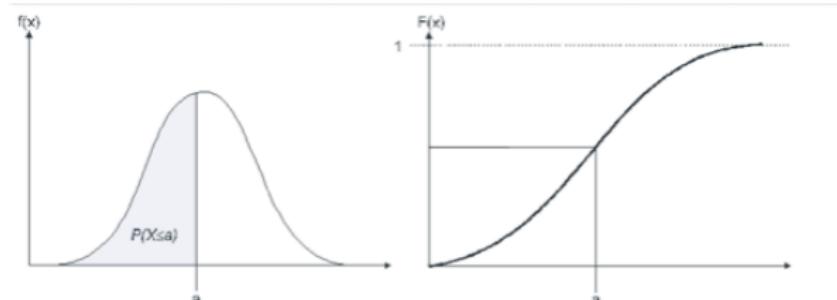
$$F(x) = \Pr(X \leq x) = \int_{-\infty}^x p(x)dx; \quad \frac{d}{dx} F(x) = p(x)$$

- Complementary cumulative distribution function (cCDF)

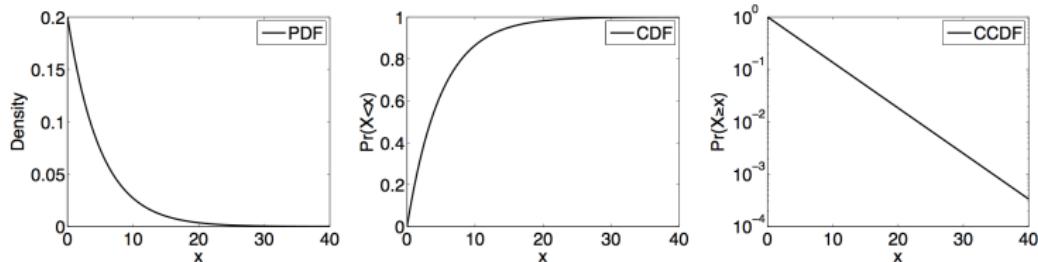
$$\bar{F}(x) = \Pr(X > x) = 1 - F(x) = \int_x^{\infty} p(x)dx$$

# Continuous distribution

- Gaussian:  $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ ,  $F(x) = \frac{1}{2}[1 + \text{erf}(\frac{x-\mu}{\sigma\sqrt{2}})]$



- Exponential ( $x \geq 0$ ):  $p(x) = \lambda e^{-\lambda x}$ ,  $F(x) = 1 - e^{-\lambda x}$ ,  $\bar{F}(x) = e^{-\lambda x}$



# Discrete distribution

- Discrete random variable  $X_i$
- Probability mass function (PMF)  $p(x)$ :

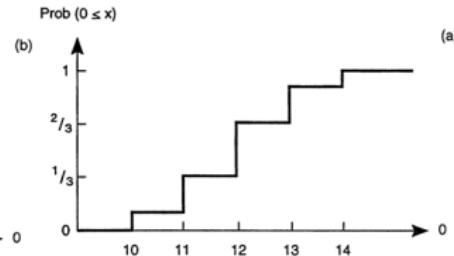
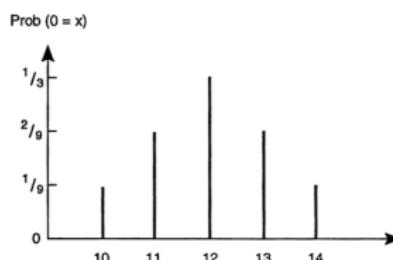
$$p(x) = \Pr(X_i = x)$$

$$p(x) \geq 0$$

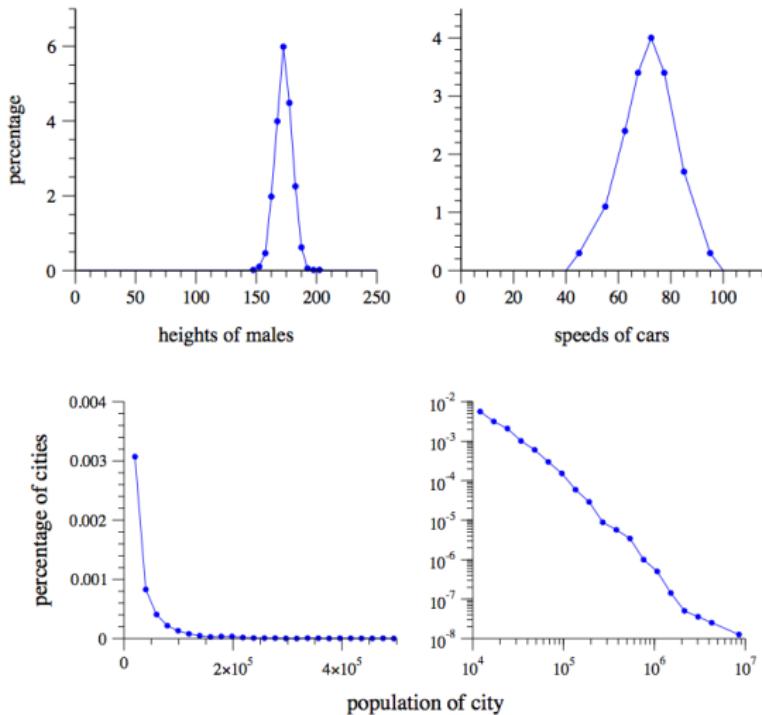
$$\sum_x p(x) = 1$$

- Cumulative distribution function (CDF)

$$F(x) = \Pr(X_i \leq x) = \sum_{x' \leq x} p(x')$$



# Empirical distributions



# Power Laws

## Continuous approximation

- Power law

$$p(x) = Cx^{-\alpha} = \frac{C}{x^\alpha}, \quad \text{for } x \geq x_{min}$$

- Normalization ( $\alpha > 1$ )

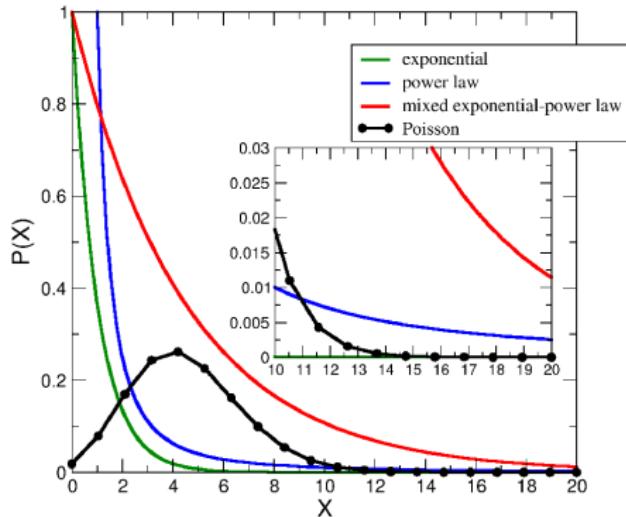
$$1 = \int_{x_{min}}^{\infty} p(x) dx = C \int_{x_{min}}^{\infty} \frac{dx}{x^\alpha} = \frac{C}{\alpha - 1} x_{min}^{-\alpha + 1}$$

$$C = (\alpha - 1)x_{min}^{\alpha - 1}$$

- Power law PDF

$$p(x) = \frac{\alpha - 1}{x_{min}} \left( \frac{x}{x_{min}} \right)^{-\alpha}$$

# Power Laws



poisson:  $p(k) = \frac{\lambda^k}{k!} e^{-\lambda}$ , exponent:  $p(x) = C e^{-\lambda x}$ , power law:  $p(x) = C x^{-\alpha}$

# Power Laws

- Power law PDF

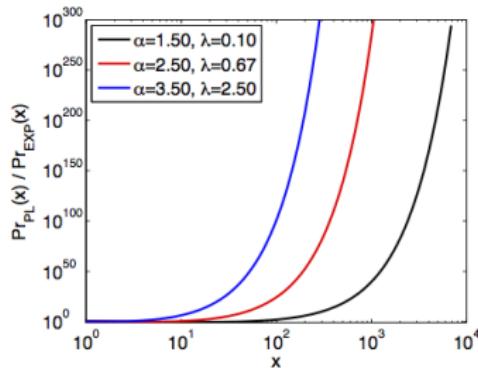
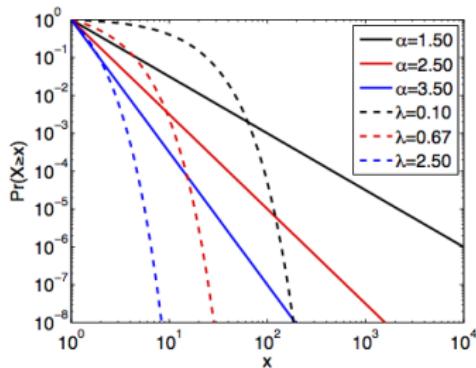
$$p(x) = Cx^{-\alpha} = \frac{\alpha - 1}{x_{min}} \left( \frac{x}{x_{min}} \right)^{-\alpha}$$

- Complimentary cumulative distribution function cCDF

$$\bar{F}(x) = Pr(X > x) = \int_x^{\infty} p(x) dx$$

$$\bar{F}(x) = \bar{C}x^{-(\alpha-1)} = \frac{C}{\alpha - 1} x^{-(\alpha-1)} = \left( \frac{x}{x_{min}} \right)^{-(\alpha-1)}$$

# Power Laws



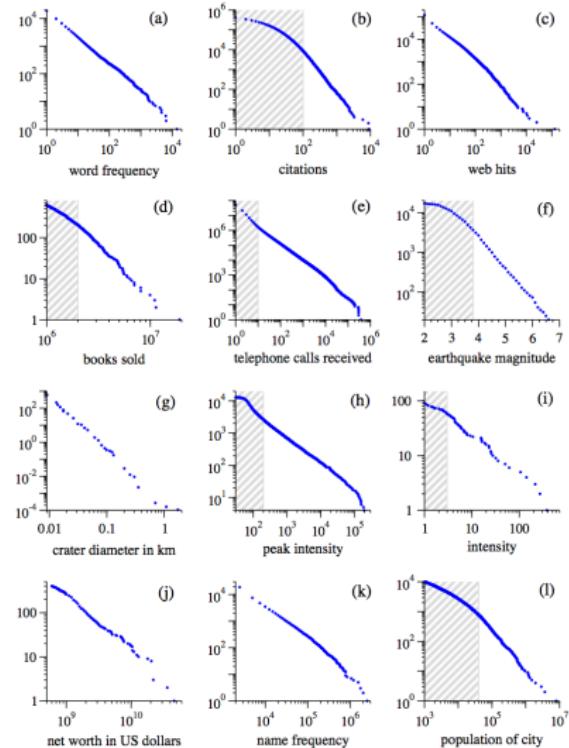
Power law:

$$p(x) = Cx^{-\alpha}, \quad \bar{F}(x) = \bar{C}x^{-(\alpha-1)}$$

$$\log p(x) = \log C - \alpha \log x, \quad \log \bar{F}(x) = \log C - (\alpha - 1) \log x$$

# Empirical distributions

log-log scale



# Moments

- PDF

$$p(x) = \frac{C}{x^\alpha}, \quad x \geq x_{\min}$$

- First moment (mean value),  $\alpha > 2$ :

$$\langle x \rangle = \int_{x_{\min}}^{\infty} x p(x) dx = C \int_{x_{\min}}^{\infty} \frac{dx}{x^{\alpha-1}} = \frac{\alpha - 1}{\alpha - 2} x_{\min}$$

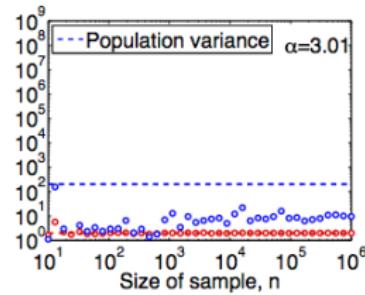
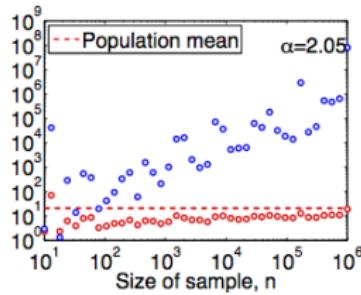
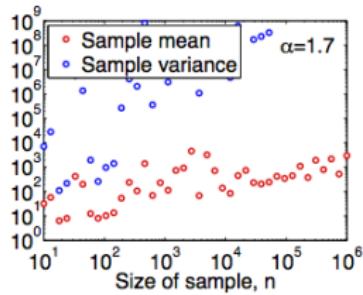
- Second moment,  $\alpha > 3$ :

$$\langle x^2 \rangle = \int_{x_{\min}}^{\infty} x^2 p(x) dx = C \int_{x_{\min}}^{\infty} \frac{dx}{x^{\alpha-2}} = \frac{\alpha - 1}{\alpha - 3} x_{\min}^2$$

- $k$ -th moment,  $\alpha > k + 1$ :

$$\langle x^k \rangle = \frac{\alpha - 1}{\alpha - 1 - k} x_{\min}^k$$

# Moments



Fisrt moment (mean):

$$\langle x \rangle = C \int_{x_{\min}}^{x_{\max}} \frac{dx}{x^{\alpha-1}} = \frac{\alpha - 1}{\alpha - 2} \left( x_{\min} - \frac{x_{\min}^{\alpha-1}}{x_{\max}^{\alpha-2}} \right)$$

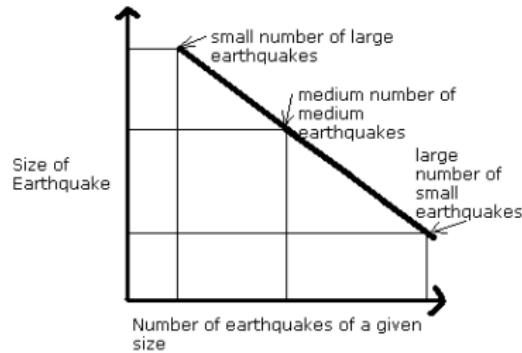
# Scale invariance

- Scaling of the density

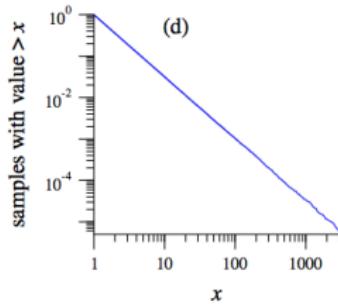
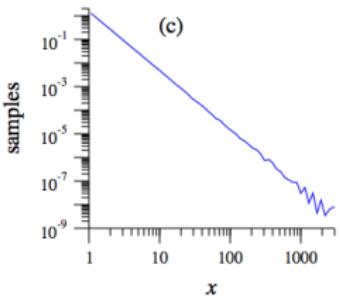
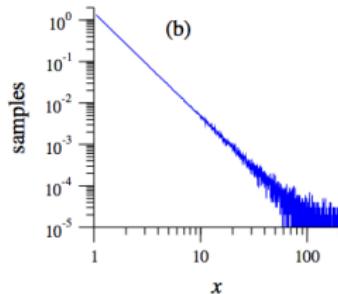
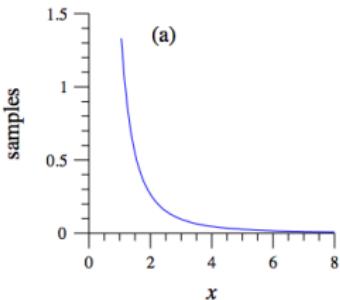
$$x \rightarrow bx, \quad p(bx) = C(bx)^{-\alpha} = b^{-\alpha} C x^{-\alpha} \propto p(x)$$

- Scale invariance

$$\frac{p(100x)}{p(10x)} = \frac{p(10x)}{p(x)}$$

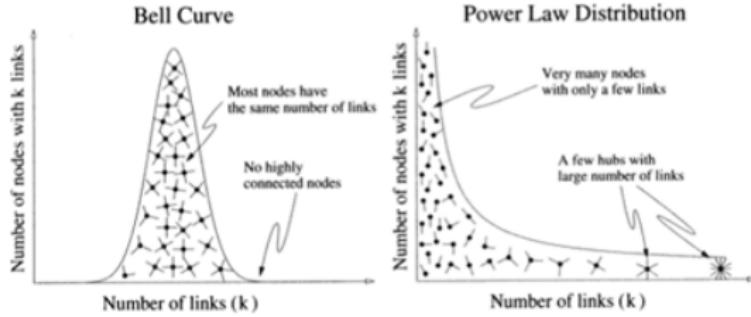


# Power law histograms



Newman et.al, 2005

# Scale-free networks



# Node degree distribution

- $k_i$  - node degree, i.e. number of nearest neighbors,  $k_i = 1, 2, \dots k_{\max}$
- $n_k$  - number of nodes with degree  $k$ ,  $n_k = \sum_i \mathcal{I}(k_i == k)$
- total number of nodes  $n = \sum_k n_k$
- Degree distribution  $P(k_i = k) \equiv P(k)$

$$P(k) = \frac{n_k}{\sum_k n_k} = \frac{n_k}{n}$$

- CDF

$$F(k) = \sum_{k' \leq k} P(k') = \frac{1}{n} \sum_{k' \leq k} n_{k'}$$

- cCDF

$$F(k) = 1 - \sum_{k' \leq k} P(k') = \frac{1}{n} \sum_{k' > k} n_{k'}$$

# Discrete power law distribution

- Power law distribution

$$P(k) = Ck^{-\gamma} = \frac{C}{k^\gamma}$$

- Normalization

$$\sum_{k=1}^{\infty} P(k) = C \sum_{k=1}^{\infty} k^{-\gamma} = C\zeta(\gamma) = 1; \quad C = \frac{1}{\zeta(\gamma)}$$

- Riemann zeta function,  $\gamma > 1$

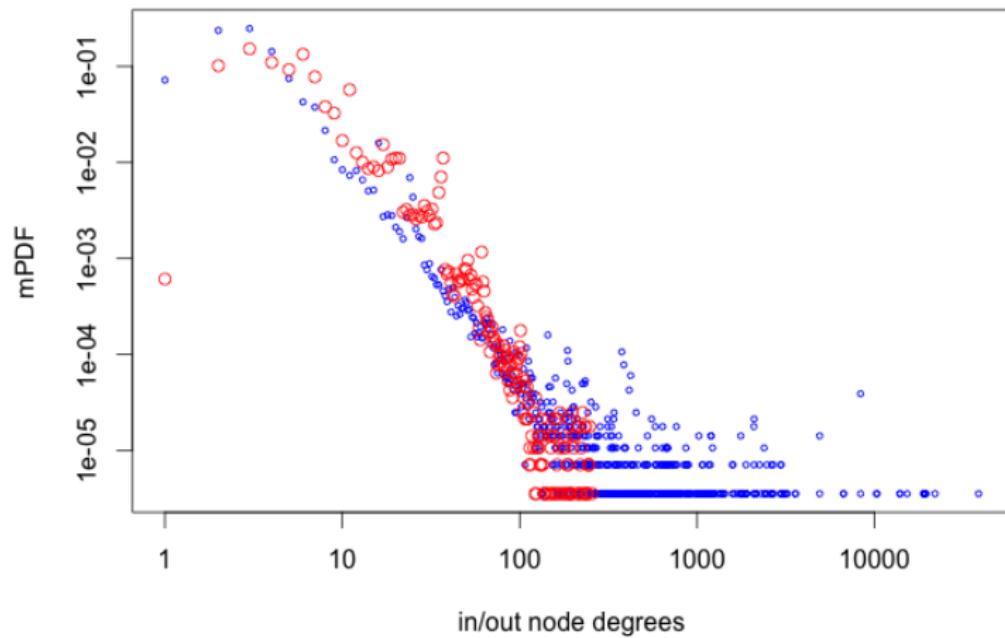
$$P(k) = \frac{k^{-\gamma}}{\zeta(\gamma)}$$

- Log-log coordinates

$$\log(P(k)) = -\gamma \log k + \log C$$

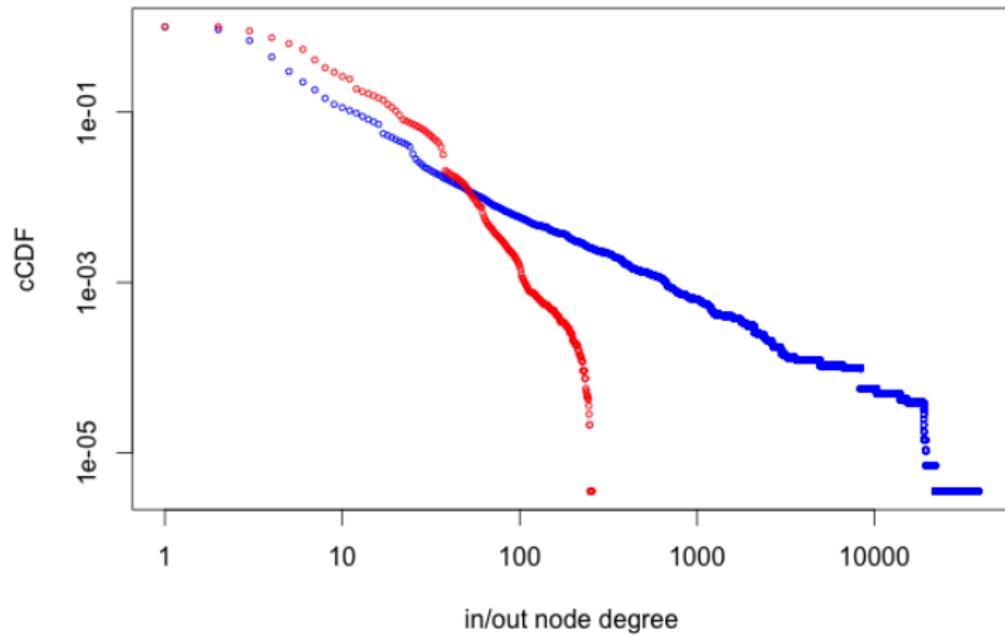
# Power law networks

## Probability mass function PMF/mPDF

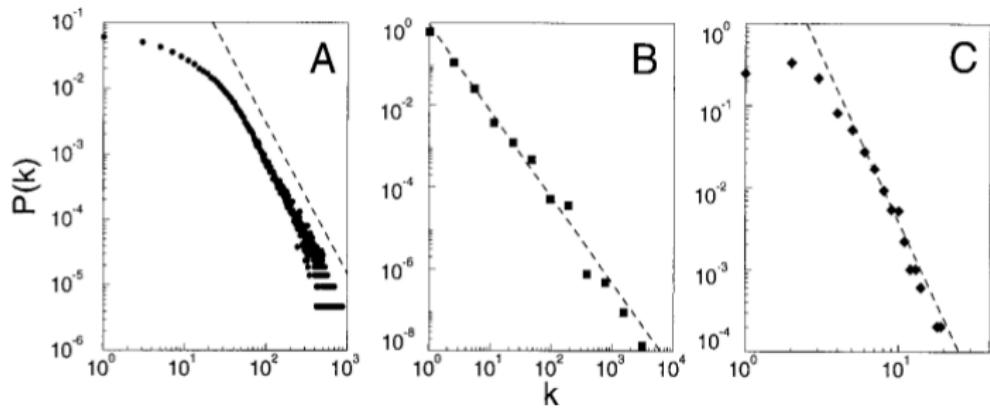


# Power law networks

## Complementary cumulative distribution function cCDF



# Power law networks



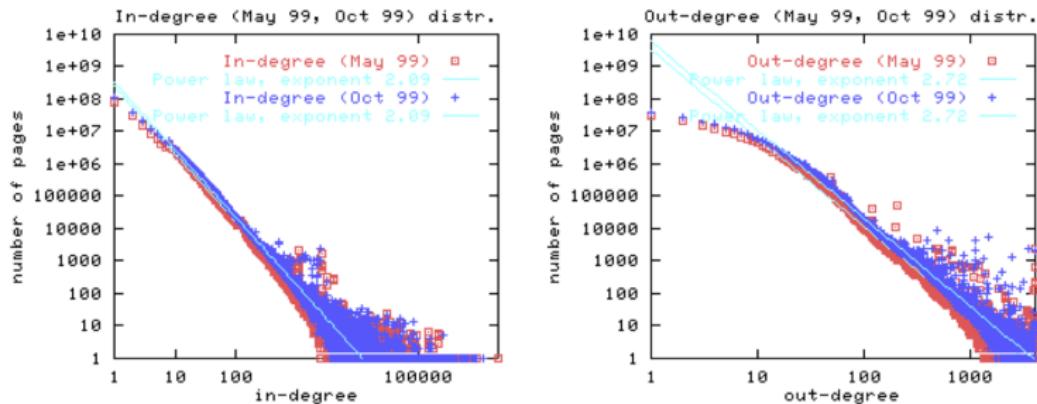
Actor collaboration graph,  $N=212,250$  nodes,  $\langle k \rangle = 28.8$ ,  $\gamma = 2.3$

WWW,  $N = 325,729$  nodes,  $\langle k \rangle = 5.6$ ,  $\gamma = 2.1$

Power grid data,  $N = 4941$  nodes,  $\langle k \rangle = 5.5$ ,  $\gamma = 4$

Barabasi et.al, 1999

# Power law networks



In- and out- degrees of WWW crawl 1999

Broder et.al, 1999

## Parameter estimation: $\alpha$

Maximum likelihood estimation of parameter  $\alpha$

- Let  $\{x_i\}$  be a set of  $n$  observations (points) independently sampled from the distribution

$$P(x_i) = \frac{\alpha - 1}{x_{\min}} \left( \frac{x_i}{x_{\min}} \right)^{-\alpha}$$

- Probability of the sample

$$P(\{x_i\}|\alpha) = \prod_i^n \frac{\alpha - 1}{x_{\min}} \left( \frac{x_i}{x_{\min}} \right)^{-\alpha}$$

- Bayes' theorem

$$P(\alpha|\{x_i\}) = P(\{x_i\}|\alpha) \frac{P(\alpha)}{P(\{x_i\})}$$

# Maximum likelihood

- log-likelihood

$$\mathcal{L} = \ln P(\alpha | \{x_i\}) = n \ln(\alpha - 1) - n \ln x_{\min} - \alpha \sum_{i=1}^n \ln \frac{x_i}{x_{\min}}$$

- maximization  $\frac{\partial \mathcal{L}}{\partial \alpha} = 0$

$$\alpha = 1 + n \left[ \sum_{i=1}^n \ln \frac{x_i}{x_{\min}} \right]^{-1}$$

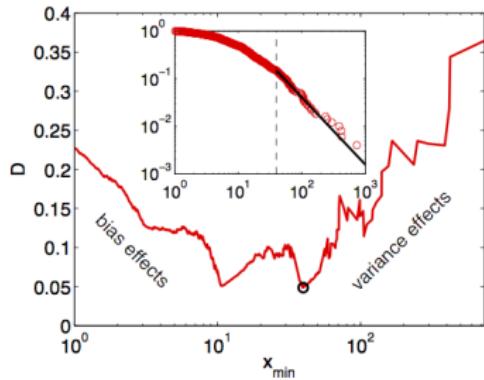
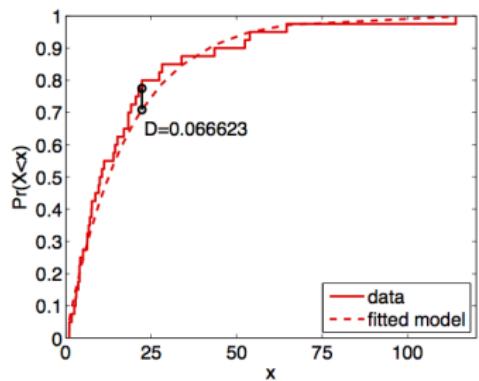
- error estimate

$$\sigma = \sqrt{n} \left[ \sum_{i=1}^n \ln \frac{x_i}{x_{\min}} \right]^{-1} = \frac{\alpha - 1}{\sqrt{n}}$$

# Parameter estimation: $x_{min}$

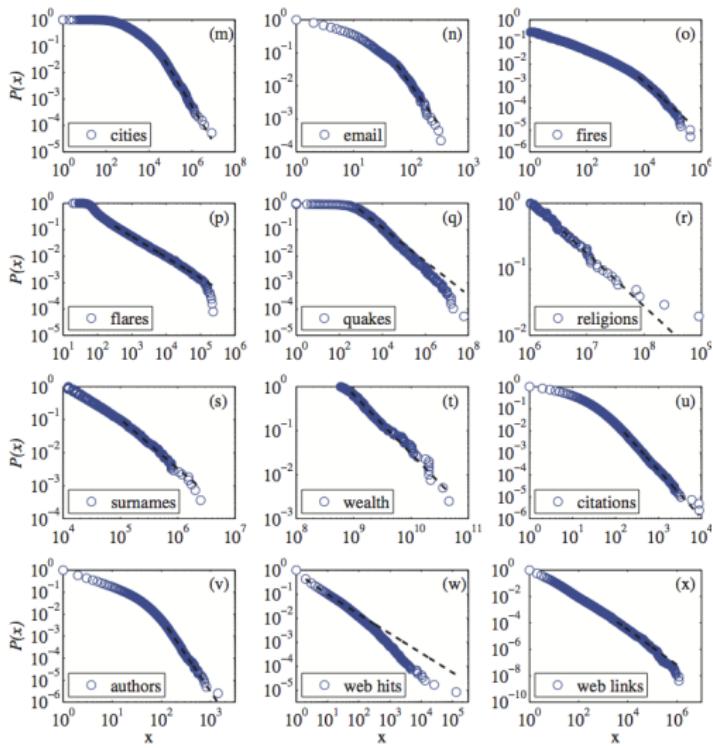
- Kolmogorov-Smirnov test (compare model and experimental CDF)

$$D = \max_x |F(x|\alpha, x_{min}) - F_{exp}(x)|$$



Clauset et.al, 2009

# Empirical models



# Word counting

Word frequency table (6318 unique words, min freq 800, corpus size > 85mln):

6187267 the

4239632 be

3093444 of

2687863 and

2186369 a

1924315 in

1620850 to

.....

801 incredibly

801 historically

801 decision-making

800 wildly

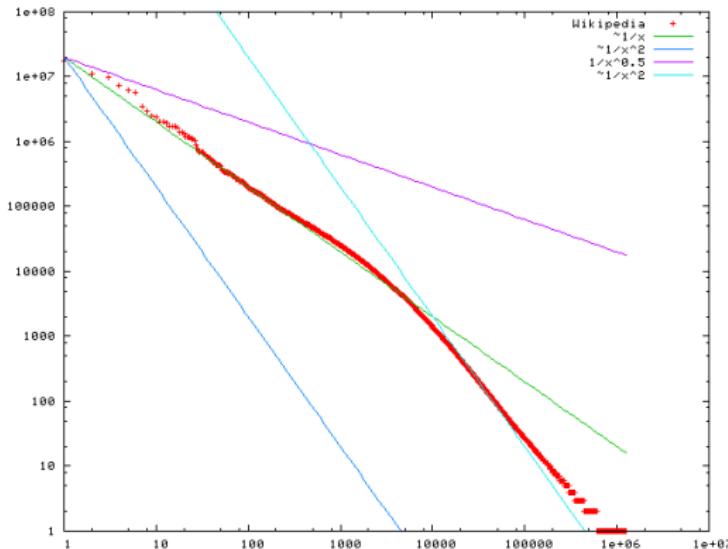
800 reformer

800 quantum

# Zips'f law

Zipf's law - the frequency of a word in an natural language corpus is inversely proportional to its rank in the frequency table  $f(k) \sim 1/k$ .

$$f(k) = \frac{1/k^s}{\sum_{k=1}^N (1/k^s)}$$



## Rank-frequency plot

- Sort items by their frequency in decreasing order (frequency table)
- Fraction of the words with frequencies higher or equal to the  $k$ -th word is cCDF  $\bar{F}(k) = \Pr(X \geq k)$ . The number of the words with frequency above  $k$ -th word is its rank  $k$ !
- Plot word rank as a function of the word frequency: rank  $k$  -  $y$  axis, frequency -  $x$  axis.
- Use rank-frequency plot instead of computing and plotting cumulative distribution of a quantity.

6187267 the

4239632 be

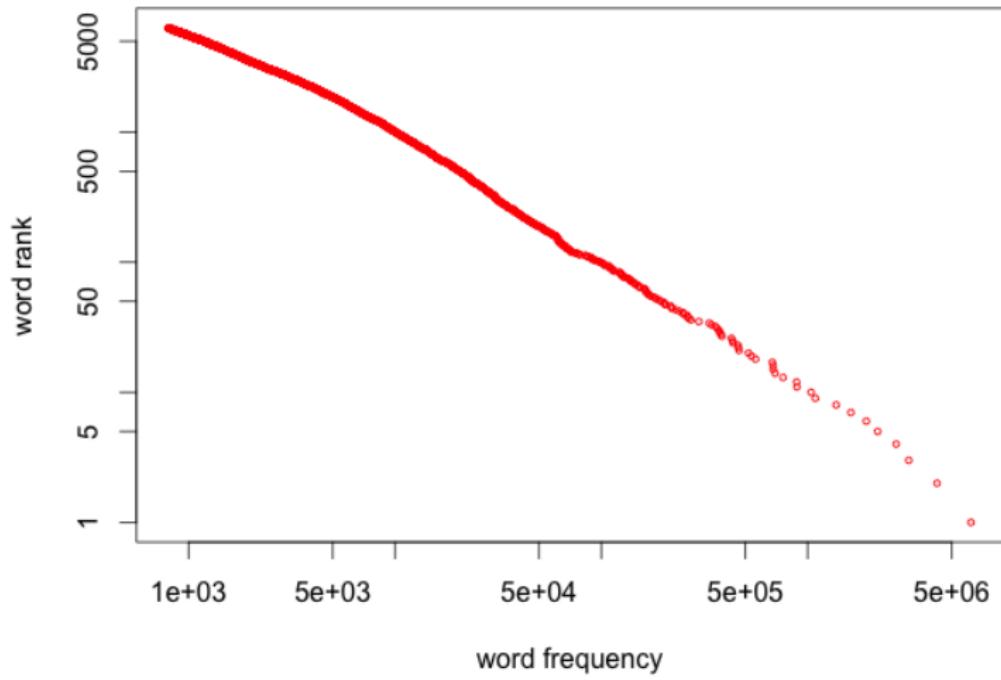
3093444 of

2687863 and

2186369 a

1924315 in

# Rank-frequency plot



## References

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- Power-Law Distribution in Empirical Data, A. Clauset, C.R. Shalizi, M.E.J. Newman, SIAM Review, Vol 51, No 4, pp. 661-703, 2009.
- A Brief History of Generative Models for Power Law and Lognormal Distributions, M. Mitzenmacher, Internet Mathematics Vol 1, No 2, pp 226-251.