Network models: dynamical growth and small world

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Network models

Empirical network features:

- Power-law (heavy-tailed) degree distribution
- Small average distance (graph diameter)
- Large clustering coefficient (transitivity)
- Giant connected component, hierarchical structure,etc

Generative models:

- Random graph model (Erdos & Renyi, 1959)
- Preferential attachement model (Barabasi & Albert, 1999)
- Small world model (Watts & Strogatz, 1998)

Motivation

Growing (evolving with time) networks:

- Citation networks
- Collaboration networks
- Web
- Social networks

Growing random graph

Stochastic growth model:

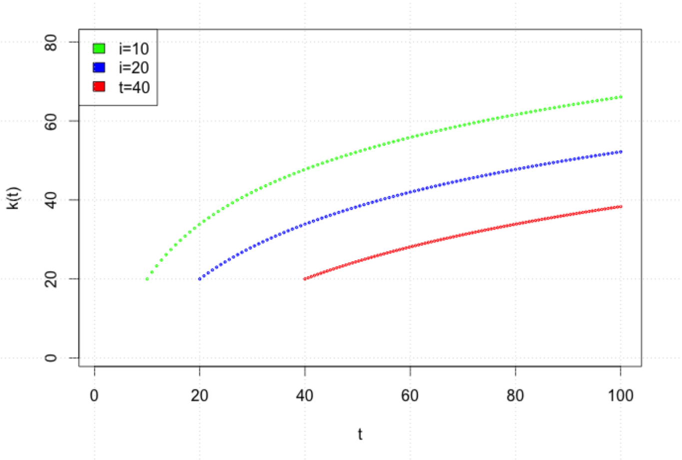
- $t = 0$, n_0 unconnected nodes
- growth: on every time step $t = \{1, 2, 3, 4...\}$ add a node with $m \le n_0$ edges $k_i(t = i) = m$
- attachment: form m edges with existing nodes uniformly at random, $\Pi(k_i) = \frac{1}{n_0+t-1}$

Expected i-node degree at t $\langle k_i(t) \rangle$:

$$k_i(t) = m + \frac{m}{n_0 + i - 1} + \frac{m}{n_0 + i} + .. \frac{m}{n_0 + t - 1}$$

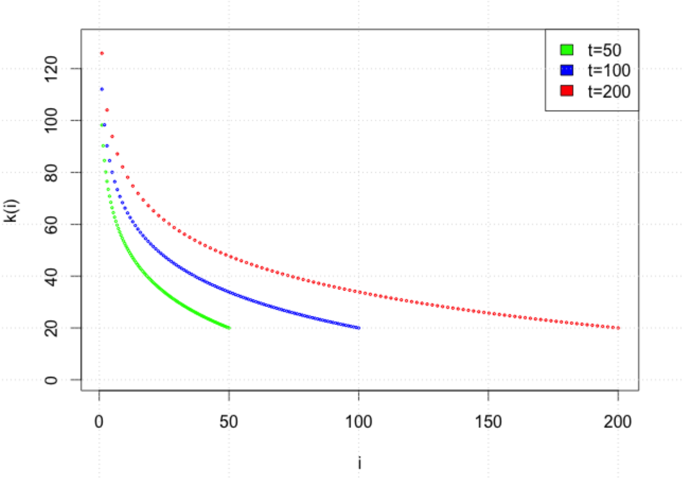
$$k_i(t) = m\left(1 + \log\left(\frac{t}{i}\right)\right)$$

Growing random graph



$$k_i(t) = m\left(1 + \log\left(\frac{t}{i}\right)\right), \quad m = 20, \ i = 10, 20, 40, \ t \geq i$$

Growing random graph



$$k_i(t) = m\left(1 + \log\left(\frac{t}{i}\right)\right), \quad m = 20, \ t = 50, 100, 200, \ i \leq t$$

Growing random graph

Find all nodes that at time t has degree less than k, $k_i(t) \le k$?
(for example $k_i(t) \le 40$)

$$
\begin{aligned}
k_i(t) = m\left(1 + \log\left(\frac{t}{i}\right)\right) &\le k \\
\log\left(\frac{t}{i}\right) &\le \frac{k}{m} - 1 \\
\frac{t}{i} &\le e^{\frac{k-m}{m}} \\
i &\ge te^{\frac{m-k}{m}}
\end{aligned}
$$

Fraction of nodes with degrees $k_i(t) \le k$ (CDF):

$$F(k) = P(k_i(t) \le k) = \frac{n_0 + t - i}{n_0 + t} = \frac{n_0 + t - te^{\frac{m-k}{m}}}{n_0 + t} \approx 1 - e^{\frac{m-k}{m}}$$

$$P(k) = \frac{d}{dk}F(k) = \frac{1}{m}e^{-\frac{k-m}{m}} = \frac{e}{m}e^{-\frac{k}{m}}, \quad k \ge m$$

Mean field approximation

Continues time approximation ($\Pi(k_i) = \frac{1}{n_0+t-1} \approx \frac{1}{t}$)

$$k_i(t + \delta t) = k_i(t) + m\Pi(k_i)\delta t$$

$$k_i(t + \delta t) = k_i(t) + \frac{m}{t}\delta t$$

Differential equation:

$$\frac{dk_i(t)}{dt} = \frac{m}{t}$$

with initial conditions: $k_i(t = i) = m$

$$\int_m^{k_i(t)} \frac{dk_i}{m} = \int_i^t \frac{dt}{t}$$

Solution:

$$k_i(t) = m\left(1 + \log\left(\frac{t}{i}\right)\right)$$

Preferential attachment model

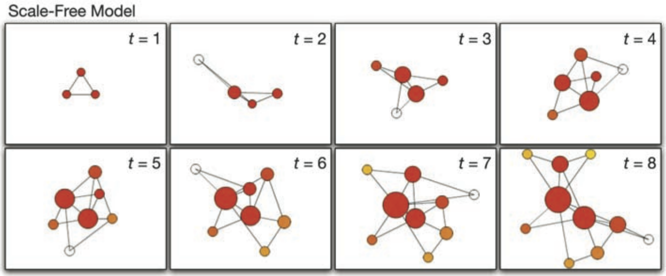
Barabasi and Albert, 1999
Dynamical growth model

- $t = 0$, n_0 nodes
- growth: on every step add a node with m edges ($m \leq n_0$), $k_i(i) = m$
- Preferential attachment: probability of linking to existing node is proportional to the node degree k_i

$$\Pi(k_i) = \frac{k_i}{\sum_i k_i}$$

after t steps: $n_0 + t$ nodes, mt edges

Preferential attachment model



Barabasi, 1999

Preferential attachment

Continues time, mean field approximation:

$$k_i(t + \delta t) = k_i(t) + m\Pi(k_i)\delta t$$

$$\frac{dk_i(t)}{dt} = m\Pi(k_i) = m\frac{k_i}{\sum_i k_i} = \frac{mk_i}{2mt}$$

$$\frac{dk_i(t)}{dt} = \frac{k_i(t)}{2t}$$

initial conditions: $k_i(i) = m$

$$\int_m^{k_i(t)} \frac{dk_i}{k_i} = \int_i^t \frac{dt}{2t}$$

Solution:

$$k_i(t) = m\left(\frac{t}{i}\right)^{1/2}$$

Preferential attachment

Time evolution of a node degree

$$k_i(t) = m\left(\frac{t}{i}\right)^{1/2}$$

Nodes with $k_i(t) \leq k$:

$$m\left(\frac{t}{i}\right)^{1/2} \leq k$$

$$i \geq \frac{m^2}{k^2}t$$

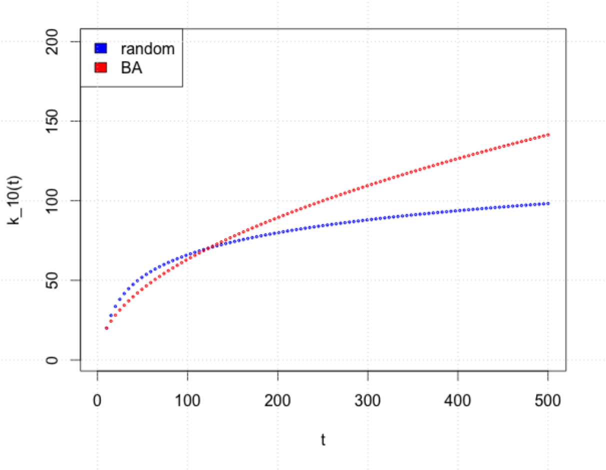
Fraction of nodes with $k_i(t) < k$ (CDF):

$$F(k) = P(k_i(t) \leq k) = \frac{n_0 + t - i}{n_0 + t} = \frac{n_0 + t - m^2 t/k^2}{n_0 + t} \approx 1 - \frac{m^2}{k^2}$$

Distribution function:

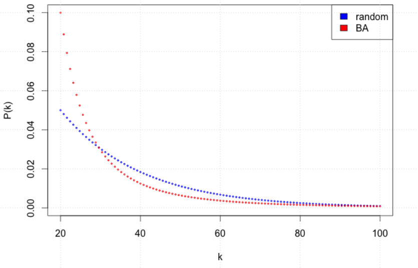
$$P(k) = \frac{d}{dk}F(k) = \frac{2m^2}{k^3}$$

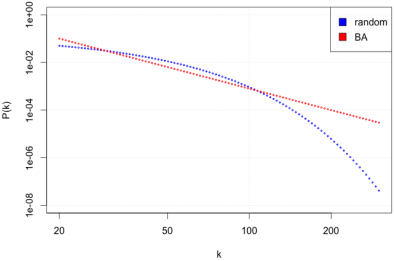
Dynamic growth



$$BA : k_i(t) = m\left(\frac{t}{i}\right)^{1/2}, \quad RG : k_i(t) = m\left(1 + \log\left(\frac{t}{i}\right)\right)$$

Dynamic growth





$$BA : P(k) = \frac{2m^2}{k^3}, \qquad RG : P(k) = \frac{e}{m} e^{-\frac{k}{m}}$$

Preferential attachment

- Power law distribution function:

$$P(k) = \frac{2m^2}{k^3}$$

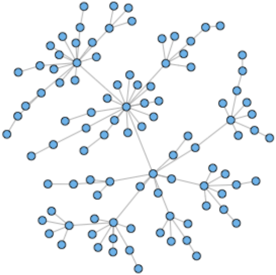
- Average path length (analytical result) :

$$\langle L \rangle \sim \log(N)/\log(\log(N))$$

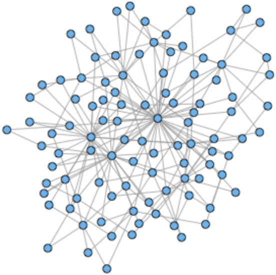
- Clustering coefficient (numerical result):

$$C \sim N^{-0.75}$$

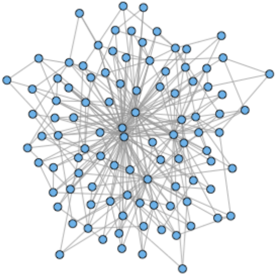
Preferential attachment model



$m = 1$



$m = 2$



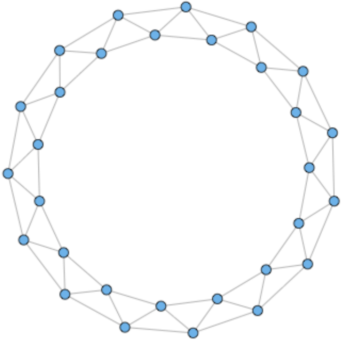
$m = 3$

Historical note

- Polya urn model, George Polya, 1923
- Yule process, Udny Yule, 1925
- Distribution of wealth, Herbert Simon,1955
- Evolution of citation networks, cumulative advantage, Derek de Solla Price, 1976
- Preferential attachment network model, Barabasi and Albert, 1999

Small world

Motivation: keep high clustering, get small diameter



Clustering coefficient $C = 1/2$
Graph diameter $d = 8$

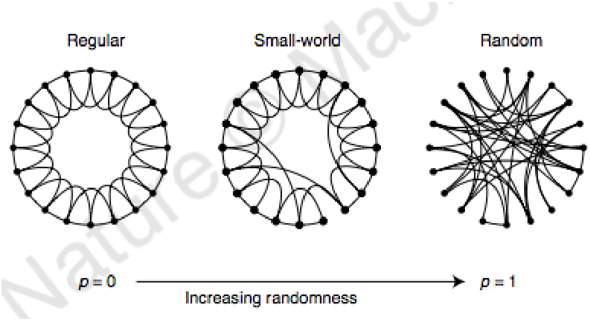
Small world

Watts and Strogatz, 1998

Single parameter model, interpolation between regular lattice and random graph

- start with regular lattice with n nodes, k edges per vertex (node degree), $k << n$
- randomly connect with other nodes with probability p, forms $pnk/2$ "long distance" connections from total of $nk/2$ edges
- $p = 0$ regular lattice, $p = 1$ random graph

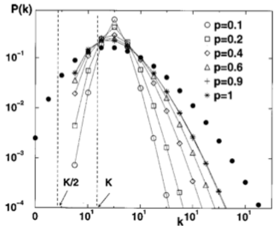
Small world

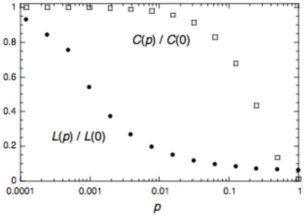


Watts, 1998

Small world model

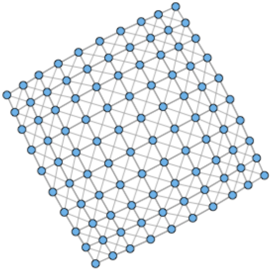
- Node degree distribution:
  Poisson like
- Ave. path length $\langle L(p) \rangle$:
  $p \to 0$, ring lattice, $\langle L(0) \rangle = 2n/k$
  $p \to 1$, random graph, $\langle L(1) \rangle = \log(n)/\log(k)$
- Clustering coefficient $C(p)$:
  $p \to 0$, ring lattice, $C(0) = 3/4 = const$
  $p \to 1$, random graph, $C(1) = k/n$

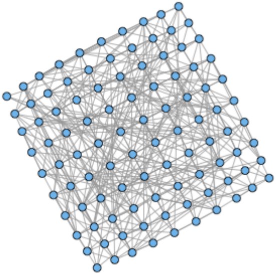




Watts, 1998

Small world model





20% rewiring:

ave. path length = 3.58 -> ave. path length = 2.32

clust. coeff = 0.49 -> clust. coeff = 0.19

Model comparison

	Random	BA model	WS model	Empirical networks
P(k)	$\frac{\lambda^k e^{-\lambda}}{k!}$	k^{-3}	poisson like	power law
C	$\langle k \rangle / N$	$N^{-0.75}$	const	large
$\langle L \rangle$	$\frac{\log(N)}{\log(\langle k \rangle)}$	$\frac{\log(N)}{\log\log(N)}$	$\log(N)$	small

References

- Collective dynamics of small-world networks. Duncan J. Watts and Steven H. Strogatz. Nature 393 (6684): 440-442, 1998
- Emergence of Scaling in Random Networks, A.L. Barabasi and R. Albert, Science 286, 509-512, 1999