

# Epidemics

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## Network Science



NATIONAL RESEARCH  
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## 1 Epidemic models

- SI model
- SIS model
- SIR model

## 2 Branching process

- Galton-Watson process

# Epidemic dynamics models

- Mathematical epidemiology
- W. O. Kermack and A. G. McKendrick, 1927
- Deterministic compartmental model (population classes)  $\{S, I, T\}$
- $S(t)$  - susceptible, number of individuals not yet infected with the disease at time  $t$
- $I(t)$  - infected, number of individuals who have been infected with the disease and are capable of spreading the disease.
- $R(t)$  - recovered, number of individuals who have been infected and then recovered from the disease, can't be infected again or to transmit the infection to others.
- Fully-mixing model
- Closed population (no birth, death, migration)
- Models: SI, SIS, SIR, SIRS,...

- $S(t)$  -susceptible ,  $I(t)$  - infected

$$S \longrightarrow I$$

$$S(t) + I(t) = N$$

- $\beta$  - infection/contact rate, number of contacts per unit time
- Infection equation:

$$I(t + \delta t) = I(t) + \beta \frac{S(t)}{N} I(t) \delta t$$

$$\frac{dI(t)}{dt} = \beta \frac{S(t)}{N} I(t)$$

- Fractions:  $i(t) = I(t)/N$ ,  $s(t) = S(t)/N$
- Equations

$$\begin{aligned}\frac{di(t)}{dt} &= \beta s(t)i(t) \\ \frac{ds(t)}{dt} &= -\beta s(t)i(t) \\ s(t) + i(t) &= 1\end{aligned}$$

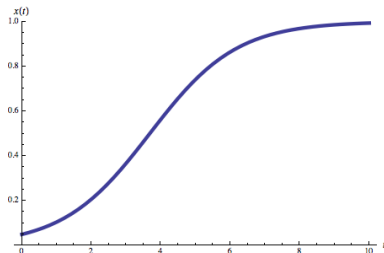
- Differential equation,  $i(t=0) = i_0$

$$\frac{di(t)}{dt} = \beta(1 - i(t))i(t)$$

# Logistic growth function

- Solution:

$$i(t) = \frac{i_0}{i_0 + (1 - i_0)e^{-\beta t}}$$



- Limit  $t \rightarrow \infty$

$$i(t) \rightarrow 1$$

$$s(t) \rightarrow 0$$

in image  $i_0 = 0.05$ ,  $\beta = 0.8$

- $S(t)$  -susceptable ,  $I(t)$  - infected,

$$S \longrightarrow I \longrightarrow S$$

$$S(t) + I(t) = N$$

- $\beta$  - infection rate (on contact),  $\gamma$  - recovery rate
- Infection equations:

$$\frac{ds}{dt} = -\beta si + \gamma i$$

$$\frac{di}{dt} = \beta si - \gamma i$$

$$s + i = 1$$

- Differential equation,  $i(t = 0) = i_0$

$$\frac{di}{dt} = (\beta - \gamma - i)i$$

- Solution

$$i(t) = \left(1 - \frac{\gamma}{\beta}\right) \frac{C}{C + e^{-(\beta-\gamma)t}}$$

where

$$C = \frac{\beta i_0}{\beta - \gamma - \beta i_0}$$

- Limit  $t \rightarrow \infty$

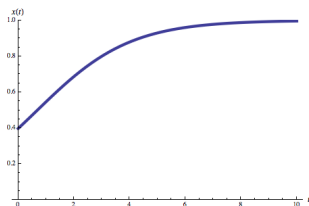
$$\beta > \gamma \quad , \quad i(t) \rightarrow \left(1 - \frac{\gamma}{\beta}\right)$$

$$\beta < \gamma \quad , \quad i(t) = i_0 e^{(\beta-\gamma)t} \rightarrow 0$$

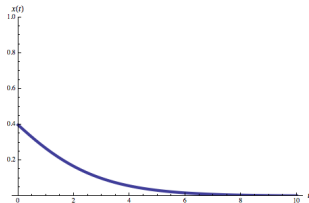


# Logistic function

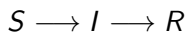
- $\beta > \gamma, \quad i(t) \rightarrow (1 - \frac{\gamma}{\beta})$



- $\beta < \gamma, \quad i(t) = i_0 e^{(\beta - \gamma)t} \rightarrow 0$



- $S(t)$  -susceptable ,  $I(t)$  - infected,  $R(t)$  - recovered



$$S(t) + I(t) + R(t) = N$$

- $\beta$  - infection rate,  $\gamma$  - recovery rate
- Infection equation:

$$\frac{ds}{dt} = -\beta si$$

$$\frac{di}{dt} = \beta si - \gamma i$$

$$\frac{dr}{dt} = \gamma i$$

$$s + i + r = 1$$

- Equation

$$\frac{ds}{dt} = -\beta s \frac{dr}{dt} \frac{1}{\gamma}$$

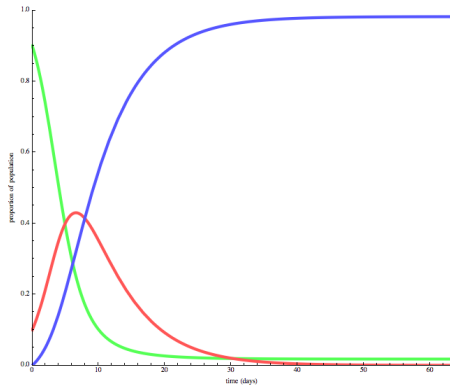
$$s = s_0 e^{-\frac{\beta}{\gamma} r}$$

$$\frac{dr}{dt} = \gamma(1 - r - s_0 e^{-\frac{\beta}{\gamma} r})$$

- Solution

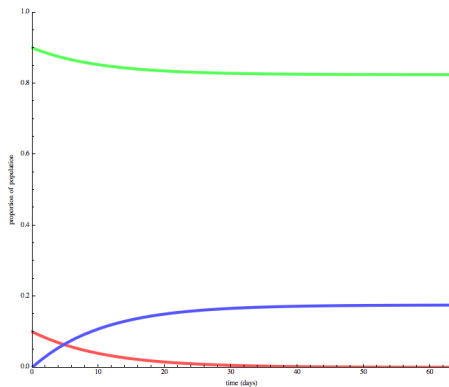
$$t = \frac{1}{\gamma} \int_0^r \frac{dr}{1 - r - s_0 e^{-\frac{\beta}{\gamma} r}}$$

# SIR model



- $\frac{\beta}{\gamma} = 4$
- $i_0 = 0.1$

# SIR model



- $\frac{\beta}{\gamma} = 0.5$
- $i_0 = 0.1$

- Equation

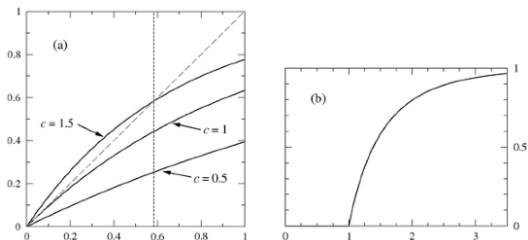
$$\frac{dr}{dt} = \gamma(1 - r - s_0 e^{-\frac{\beta}{\gamma} r})$$

- Limits:  $t \rightarrow \infty$ ,  $\frac{dr}{dt} = 0$ ,  $r_\infty = \text{const}$ ,

$$1 - r_\infty = s_0 e^{-\frac{\beta}{\gamma} r_\infty}$$

- Initial conditions:  $r(0) = 0$ ,  $i(0) = c/N$ ,  $s(0) = 1 - c/N \approx 1$

$$1 - r_\infty = e^{-\frac{\beta}{\gamma} r_\infty}$$



$$r_\infty = 1 - e^{-R_0 r_\infty}, \quad R_0 = \frac{\beta}{\gamma}$$

$$(r_\infty)'|_{r_\infty=0} = (1 - e^{-R_0 r_\infty})'|_{r_\infty=0},$$

critical point:  $R_0 = 1$

- $r_\infty$  - the total size of the outbreak
- Epidemic threshold

Epidemics:  $R_0 > 1$ ,  $\beta > \gamma$  ,  $r_\infty = \text{const} > 0$

No epidemics:  $R_0 < 1$ ,  $\beta < \gamma$  ,  $r_\infty \rightarrow 0$

- $\beta$  - infection rate,  $\gamma$  - recovery rate  $\rightarrow 1/\gamma$  - average time to recover
- Basic reproduction number

$$R_0 = \frac{\beta}{\gamma}$$

It is average number of people infected by a person before his recovery

$$R_0 = E[\beta\tau] = \beta \int_0^\infty \gamma\tau e^{-\gamma\tau} d\tau = \frac{\beta}{\gamma}$$



# Model of contagion

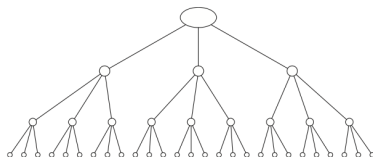
Simple model of contagion (decease transmission)

- 1st-wave: first infected person enters the population and transmits to each person he meets with probability  $p$ . Suppose he meets  $k$  people while contagious
- 2nd-wave: Each infected person from 1st wave meets  $k$  new people and independently transmits infection with probability  $p$
- 3rd-wave: ....

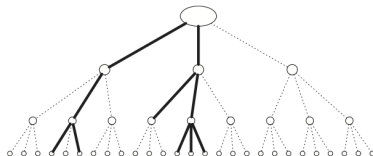
This is Galton-Watson branching stochastic process (Proposed by Francis Galton 1889 as a model for extinction of family names)

Population is organized as a tree

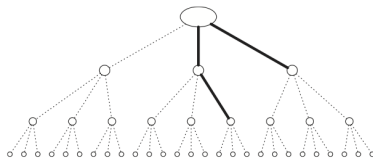
# Branching process



(a)



(b)



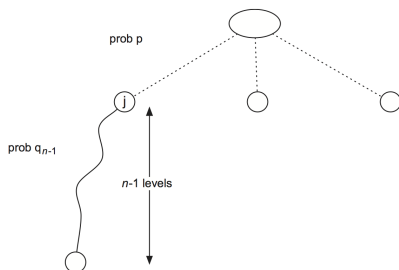
(c)

image from David Easley, Jon Kleinberg, 2010

# Branching process

## Extinction probability

- let  $q_n$  - probability that infection persists  $n$  steps (levels of the tree)
- $pq_{n-1}$  - probability that spreads through one first contact and then survives  $n - 1$  levels



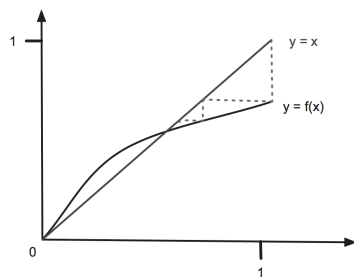
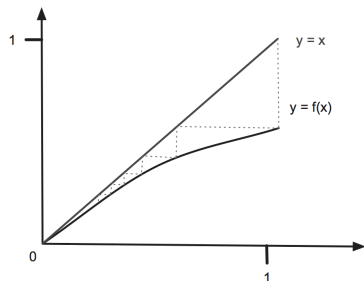
- $(1 - pq_{n-1})^k$  - probability that will not spread through any of the subtrees

$$(1 - pq_{n-1})^k = 1 - q_n$$

# Branching process

- Recurrence ( $q_n$  - probability that infection persists through  $n$  steps)

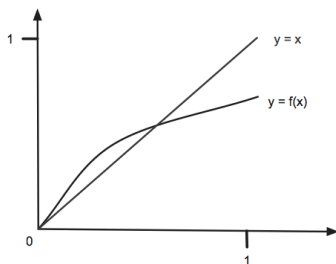
$$q_n = 1 - (1 - pq_{n-1})^k, \quad q_0 = 1$$



# Branching process

- limiting probability  $q^* = \lim_{n \rightarrow \infty} q_n$

$$q^* = 1 - (1 - pq^*)^k$$



- Slope:

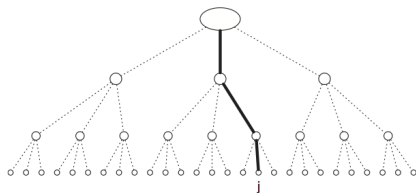
$$pk(1 - pq)^{k-1} \Big|_{q=0} = 1$$

- When  $R_0 = pk > 1$ , there is a non zero probability of infection persists

# Branching process

Galton-Watson branching random process:

- if  $R_0 = 1$ , the mean of number of infected nodes does not change
- if  $R_0 > 1$ , the mean grows geometrically as  $R_0^n$
- if  $R_0 < 1$ , the mean shrinks geometrically as  $R_0^n$



$R_0 = 1$  - point of phase transition

- A Contribution to the Mathematical Theory of Epidemics. , Kermack, W. O. and McKendrick, A. G. , Proc. Roy. Soc. Lond. A 115, 700-721, 1927.
- The Mathematics of Infectious Disease, Herbert W. Hethcote, SIAM Review, Vol. 42, No. 4, p. 599-653, 2000