Social learning

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Lecture outline

1. Social learning
2. Reaching consensus
   - DeGroot model
3. Social influence networks

Social learning

- Social learning is changing ones behavior or beliefs based on direct observation of others (imitation, aggregation, adoptation)
- Local interactions (network)
- Information is dispersed through the network
- No centralized mechanism for information aggregation

Reaching a consensus

"Reaching a consensus", Morris DeGroot 1974

Consensus - mutual agreement on a subject among group of people

- Group of people with opinions on the subject
- Can a common belief be reached?
- How long would it take?
- How each individual belief contribute to consensus?
- Which individuals have the most influence over final beliefs?

DeGroot Model

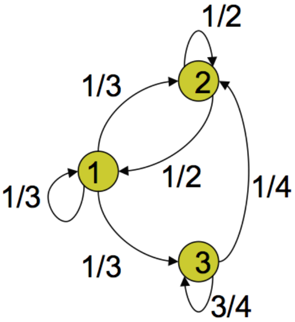
- Opinion $p_i(t) \in [0..1]$,
- Initial opinion on the subject $p_i(0)$
- T_{ij} is weight on the opinion of others, $i \to j$, how much i "listens to opnion" of j, $\sum_j T_{ij} = 1$
- Opinion update

$$p_i(t+1) = \sum_j T_{ij} p_j(t)$$

- Could a consensus be reached, i.e. all opinions converge to the same value?

$$\lim_{t \to \infty} p_i(t) = p^{\infty}$$

Example 1



$$T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

Example 1

- Initial beliefs

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Updating

$$p(1) = Tp(0) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix}$$

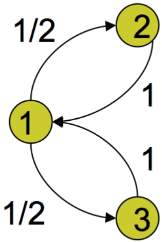
$$p(2) = Tp(1) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5/18 \\ 5/12 \\ 1/8 \end{pmatrix}$$

- Consensus

$$p(20) = Tp(19) = \begin{pmatrix} 3/11 \\ 3/11 \\ 3/11 \end{pmatrix}$$

$$p(21) = Tp(20) = p(20)$$

Example 2



$$T = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Example 2

- Initial beliefs

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Updating

$$p(1) = Tp(0) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$p(2) = Tp(1) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Consensus

$$\nexists \lim_{t \to \infty} T^t p(0)$$

Perron - Frobenius Theorem

Perron & Frobenius, 1912, linear algebra.
For stochastic matrices $\sum_j T_{ij} = 1$:

Theorem

Let $\mathbf{T}$ *be a square 1) non-negative* $T_{ij} \geq 0$, *2) irreducible, 3) aperiodic stochastic matrix. Then there are exist*

$$\lim_{t \to \infty} T_{ij}^t = \pi_j$$

where

$$\pi_j = \sum_i \pi_i T_{ij}$$

$\pi = (\pi_1, \pi_2, ..\pi_n)$ *- is the left eigenvalue of* $\mathbf{T}$, *corresponding to* $\lambda_1 = 1$ *and* $\sum_i \pi_i = 1$

Irreducible matrix - strongly connected graph
Aperiodic matrix - the greatest common divisor of the length of the cycles in the associated graph is $gcd = 1$

Limiting belief

- Limiting belief

$$p(t) = Tp(t-1) = T^2 p(t-2) = T^t p(0)$$

$$p^\infty = \lim_{t\to\infty} p(t) = \lim_{t\to\infty} T^t p(0)$$

$$\lim_{t\to\infty} T^t = \lim_{t\to\infty} \begin{pmatrix} T_{11} & \dots & T_{1n} \\ \dots & \dots & \dots \\ T_{n1} & \dots. & T_{nn} \end{pmatrix}^t = \begin{pmatrix} \pi_1 & \pi_2 & \dots & \pi_n \\ \pi_1 & \pi_2 & \dots & \pi_n \\ \pi_1 & \pi_2 & \dots & \pi_n \end{pmatrix}$$

$$\mathbf{p}^\infty = \begin{pmatrix} \pi_1 & \pi_2 & \dots & \pi_n \\ \pi_1 & \pi_2 & \dots & \pi_n \\ \pi_1 & \pi_2 & \dots & \pi_n \end{pmatrix} \begin{pmatrix} p_1(0) \\ \dots \\ p_n(0) \end{pmatrix} = \begin{pmatrix} p^\infty \\ \dots \\ p^\infty \end{pmatrix}$$

- Left eigenvector

$$\pi \mathbf{T} = \pi \lambda$$

Limiting belief

- Example 1

$$T^{20} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}^{20} = \begin{pmatrix} 0.27 & 0.36 & 0.36 \\ 0.27 & 0.36 & 0.36 \\ 0.27 & 0.36 & 0.36 \end{pmatrix}$$

$$p^{\infty} = \begin{pmatrix} 0.27 & 0.36 & 0.36 \\ 0.27 & 0.36 & 0.36 \\ 0.27 & 0.36 & 0.36 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.27 \\ 0.27 \\ 0.27 \end{pmatrix}$$

- Left eigenvector

$$\pi \mathbf{T} = \pi \lambda$$

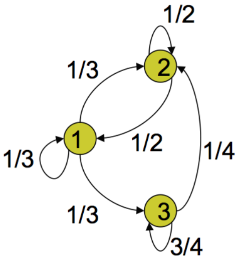
$$\lambda = \{1, 0.5, 0.083\}, \quad \pi_1 = (0.27, 0.36, 0.36)$$

- Limiting belief

$$p^{\infty} = \lim_{t\to\infty} T^t p(0) = \Pi p(0) = \sum_i \pi_i p_i(0)$$

- Influence vector

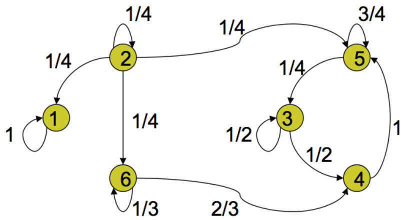
$$\pi = (\pi_1, \pi_2, ..\pi_n)$$



$\pi = (0.27, 0.36, 0.36)$

Closed sets

- A set of nodes C is called a *closed set* if there is no direct link from the node in C to the node outside C (there is no $i \in C$ and $j \notin C$ such that $T_{ij} > 0$)



- T is convergent if and only if every set of nodes that is strongly connected and closed is aperiodic
- Every strongly connected closed and aperiodic set will reach own consensus

Model extensions

- Time-varying updates:

$$p(t) = \left[(1-\lambda(t))I + \lambda(t)\hat{T}\right] p(t-1)$$

DeMarzo, Vayanos, Zwibel, 2003

- Similar beliefs:

$$T_{ij}(p(t)) = \begin{cases} \frac{1}{n_i(p(t))}, & |p_i(t) - p_j(t)| < d, n_i = \#\{k : |p_i - p_k| < d\} \\ 0, & otherwise \end{cases}$$

Krause

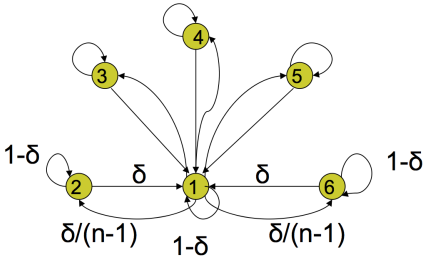
- Close beliefs:

$$T_{ij}(p(t)) = \frac{e^{-\gamma_{ij}|p_i(t)-p_j(t)|}}{\sum_k e^{-\gamma_{ik}|p_i(t)-p_k(t)|}}$$

- Under some conditions consensus could be reached in all models

Wisdom of the crowd

- Wisdom of crowd - taking into account collective opinion of a group of individuals for collective decision
- Claim: collective decision is better that decision by any individual due to independent judgements and aggregation process that removes random noise and averages out decisions.
- Will consensus converge to "correct" values?
- Opinion leader



Golub and Jackson, 2007

Social influence networks

Noah Friedkin 1991, ..,1999

- interpersonal influence for collective decisions
- modifying attitudes and opinions by interaction
- influence process in a group of N actors:

$$\mathbf{y}(t) = \mathbf{DTy}(t-1) + (\mathbf{I} - \mathbf{D})\mathbf{y}(0)$$

  $\mathbf{y}(t)$ - vector of actors' opinion at time t
  $\mathbf{y}(0)$ - vector of actors' initial opinion
  $\mathbf{T}$ - N x N matrix of interpersonal influence
  $\mathbf{D} = diag(d_{11}, .., d_{NN})$ - matrix of actors' susceptibilities to interpersonal influence

- do iterations converge?

Limiting beliefs

- Assuming the process reaches an equilibrium, $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{y}^\infty$

$$\mathbf{y}^\infty = \mathbf{D}\mathbf{T}\mathbf{y}^\infty + (\mathbf{I} - \mathbf{D})\mathbf{y}(0)$$

- Solution:

$$\mathbf{y}^\infty = (\mathbf{I} - \mathbf{D}\mathbf{T})^{-1}(\mathbf{I} - \mathbf{T})\mathbf{y}(0)$$

- Limiting beliefs (no consensus):

$$\mathbf{y}^\infty = \begin{pmatrix} y_1^\infty \\ y_2^\infty \\ \dots \\ y_n^\infty \end{pmatrix}$$

References

- Reaching a Consensus, M. DeGroot, J. Amer. Stat. Assoc., Vol 69, N 345, pp 118-121, 1974
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- Naive Learning in Social Networks and the Wisdom of Crowds, B. Golub and M. Jackson, Amer. Econ. J. Microeconomics. 2010
- Social Influence Networks and Opinion Change, Friedkin, Noah E. and Eugene C. Johnsen. Advances in Group Processes 16:1-29, 1999