# Network models: random graphs

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## Network Science

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# Network models

Empirical network features:

- Power-law (heavy-tailed) degree distribution
- Small average distance (graph diameter)
- Large clustering coefficient (transitivity)
- Giant connected component, hierarchical structure,etc

Generative models:

- Random graph model (Erdos & Renyi, 1959)
- "Small world" model (Watts & Strogatz, 1998)
- Preferential attachement model (Barabasi & Albert, 1999)

# Random graph model

Graph $G\{E, V\}$, nodes $n = |V|$, edges $m = |E|$
Erdos and Renyi, 1959.
Random graph models

- $G_{n,m}$, a randomly selected graph from the set of $C_N^m graphs, N = \frac{n(n-1)}{2}$ , with $n$ nodes and $m$ edges
- $G_{n,p}$, each pair out of $N = \frac{n(n-1)}{2}$ pairs of nodes is connected with probability $p$, $m$ - random number

$$\langle m \rangle = p\frac{n(n-1)}{2}$$

$$\langle k \rangle = \frac{1}{n}\sum_i k_i = \frac{2\langle m \rangle}{n} = p\ (n-1) \approx pn$$

$$\rho = \frac{\langle m \rangle}{n(n-1)/2} = p$$

# Random graph model

- Probability that $i$-th node has a degree $k_i = k$

$$P(k_i = k) = P(k) = C_{n-1}^{k} p^k (1-p)^{n-1-k}$$

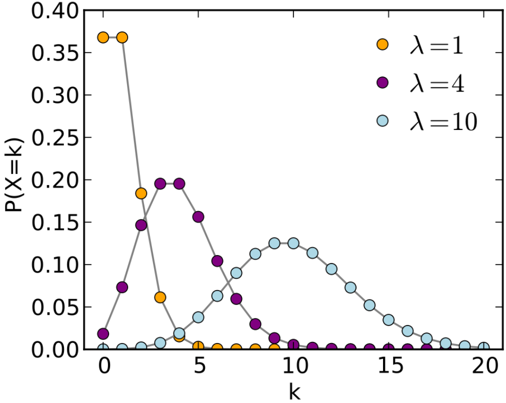
(Bernoulli distribution)
$p^k$ - probability that connects to $k$ nodes (has $k$-edges)
$(1-p)^{n-k-1}$ - probability that does not connect to any other node
$C_{n-1}^{k}$ - number of ways to select $k$ nodes out of all to connect to

- Limiting case of Bernoulli distribution, when $n \to \infty$ at fixed $\langle k \rangle = pn = \lambda$

$$P(k) = \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!} = \frac{\lambda^k e^{-\lambda}}{k!}$$

(Poisson distribution)

# Poisson Distribution



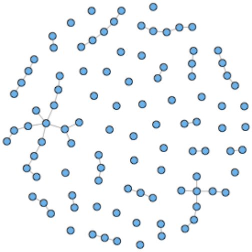

$$P(k_i = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad \lambda = pn$$

# Phase transition

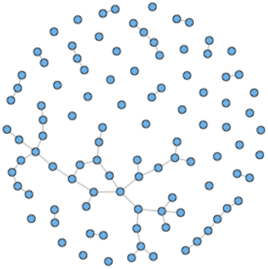
Consider $G_{n,p}$ as a function of $p$

- $p = 0$, empty graph
- $p = 1$, complete (full) graph
- There are exist critical $p_c$, structural changes from $p < p_c$ to $p > p_c$
- Gigantic connected component appears at $p > p_c$

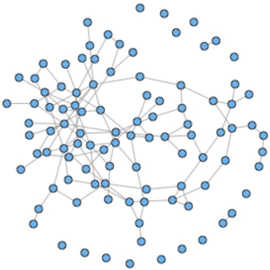
# Random graph model



$p < p_c$

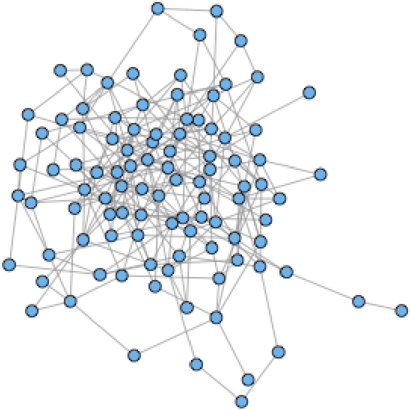


$p = p_c$



$p > p_c$

# Random graph model



$$p >> p_c$$

# Phase transition

Let $u$ fraction of nodes that do not belong to GCC. The probability that a node does not belong to GCC

$$u = P(k=0) + P(k=1)\cdot u + P(k=2)\cdot u^2 + P(k=3)\cdot u^3 ... =$$
$$= \sum_{k=0}^{\infty} P(k)u^k = \sum_{k=0} \frac{\lambda^k e^{-\lambda}}{k!} u^k = e^{-\lambda} e^{\lambda u} = e^{\lambda(u-1)}$$

Let $s$ -fraction of nodes belonging to GCC (size of GCC)

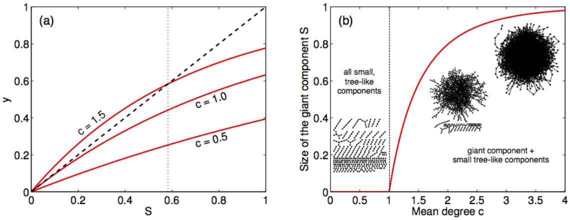
$$s = 1 - u$$

$$1 - s = e^{-\lambda s}$$

when $\lambda \to \infty, \quad s \to 1$
when $\lambda \to 0, \quad s \to 0$
$(\lambda = pn)$

# Phase transition

$$s = 1 - e^{-\lambda s}$$




non-zero solution exists when (at $s = 0$):

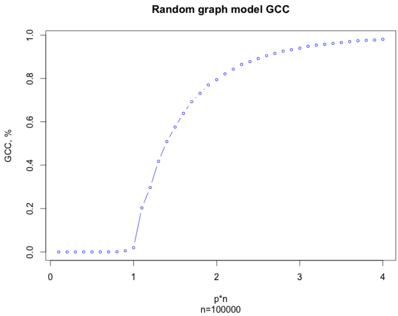
$$\lambda e^{-\lambda s} > 1$$

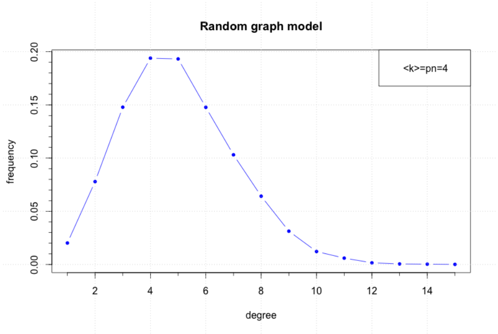
critical value:

$$\lambda_c = 1$$

$$\lambda_c = \langle k \rangle = p_c n = 1, \quad p_c = \frac{1}{n}$$

# Numerical simulations




$$\langle k \rangle = pn$$

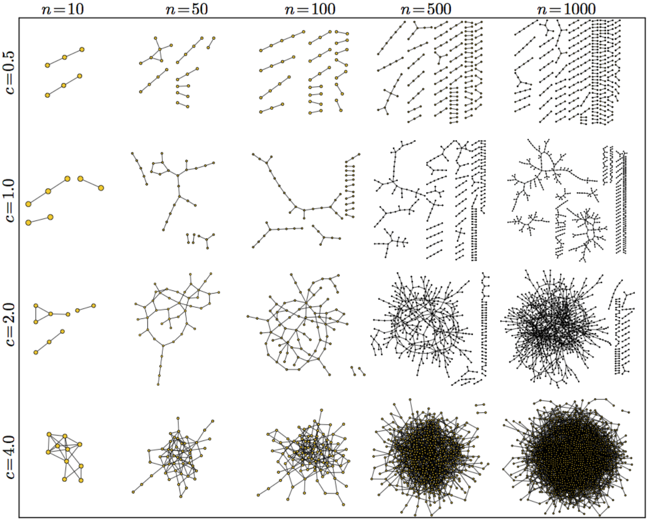
# Phase transition

Graph $G(n, p)$, for $n \to \infty$, critical value $p_c = 1/n$

- when $p < p_c$, ($\langle k \rangle < 1$) there is no components with more than $O(\ln n)$ nodes, largest component is a tree
- when $p = p_c$, ($\langle k \rangle = 1$) the largest component has $O(n^{2/3})$ nodes
- when $p > p_c$, ($\langle k \rangle > 1$) gigantic component has all $O(n)$ nodes

Critical value: $\langle k \rangle = p_c n = 1$- on average one neighbor for a node

# Phase transition

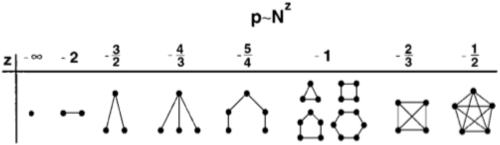



Clauset, 2014

# Threshold probabilities

Graph $G(n, p)$

Threshold probabilities when different subgraphs of $k$-nodes and $l$-edges appear in a random graph $p_c \sim n^{-k/l}$




When $p > p_c$:

- $p_c \sim n^{-k/(k-1)}$, having a tree with $k$ nodes
- $p_c \sim n^{-1}$, having a cycle with $k$ nodes
- $p_c \sim n^{-2/(k-1)}$, complete subgraph with $k$ nodes

Barabasi, 2002

# Clustering coefficient

- Clustering coefficient (probability that two neighbors link to each other):

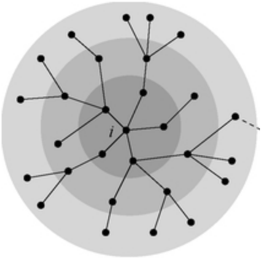
$$C_i(k) = \frac{\#\text{of links between NN}}{\#\text{max number of links NN}} = \frac{pk(k-1)/2}{k(k-1)/2} = p$$

$$C = p = \frac{\langle k \rangle}{n}$$

- when $n \to \infty, \quad C \to 0$

# Graph diameter

- $G(n, p)$ is locally tree-like (GCC) (no loops; low clustering coefficient)




- on average, the number of nodes $d$ steps away from a node

$$n = 1 + \langle k \rangle + \langle k \rangle^2 + ... \langle k \rangle^D = \frac{\langle k \rangle^{D+1} - 1}{\langle k \rangle - 1} \approx \langle k \rangle^D$$

- in GCC, around $p_c$ , $\langle k \rangle^D \sim n$,

$$D \sim \frac{\ln n}{\ln \langle k \rangle}$$

# Random graph

$G(n, p)$ model:

- Node degree distribution - Poisson:

$$P(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad \lambda = pn$$

- Clustering coefficient - small:

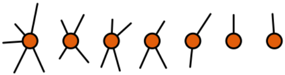
$$C = p$$

- Graph diameter - small:

$$D \sim \ln n$$

# Configuration model

- Random graph with $n$ nodes with a given degree sequence: $D = \{k_1, k_2, k_3..k_n\}$ and $m = 1/2 \sum_i k_i$ edges.
- Construct by randomly matching two stubs and connecting them by an edge.



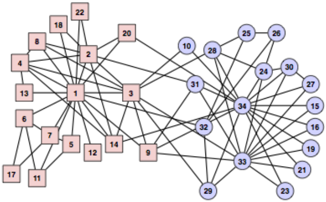
- Can contain self loops and multiple edges
- Probability that two nodes $i$ and $j$ are connected

$$p_{ij} = \frac{k_i k_j}{2m - 1}$$

- Will be a simple graph for special "graphical degree sequence"

# Configuration model

Can be used as a "null model" for comparative network analysis

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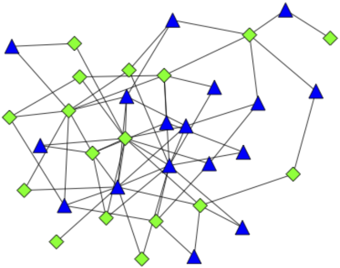

configuration model

Clauset, 2014

# References


- On random graphs I, P. Erdos and A. Renyi, Publicationes Mathematicae 6, 290297 (1959).
- On the evolution of random graphs, P. Erdos and A. Renyi, Publicaton of the Mathematical Institute of the Hungarian Academy of Sciences, 17-61 (1960)