Centrality Measures

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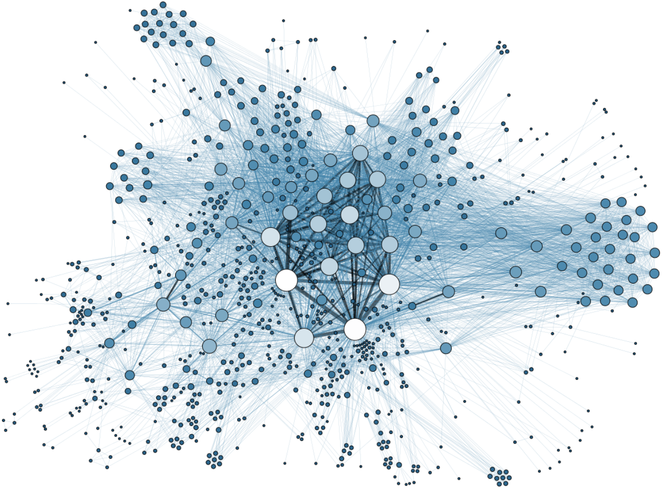
Lecture outline

1. Notion of centrality
2. Graph-theoretic measures
3. Node centralities
   - Degree centrality
   - Closeness centrlity
   - Betweenness centrality
   - Eigenvector centrality
   - Katz and Bonacich centralities
4. Rank correlation

Cetrality

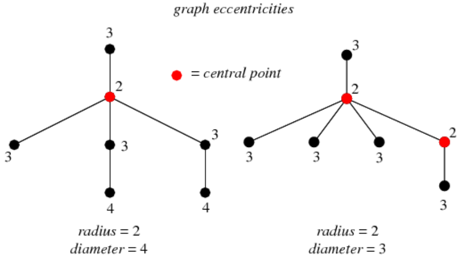
Which vertices are important?



M.Grandjean, 2014

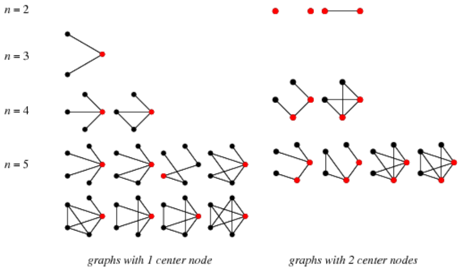
Graph-theoretic measures

- The **eccentricity** $\epsilon(v)$ of a vertex v is the maximum distance between v and any other vertex u of the graph $\epsilon(v) = \max_{u \in V} d(u, v)$
- Graph **diameter** is the maximum eccentricity $d = \max_{v \in V} \epsilon(v)$
- Graph **radius** is the minimum eccentricity $r = \min_{v \in V} \epsilon(v)$.
- A point v is a **central point** of a graph if the eccentricity of the point equals the graph radius $\epsilon(v) = r$



from Eric Weisstein *MathWorld*

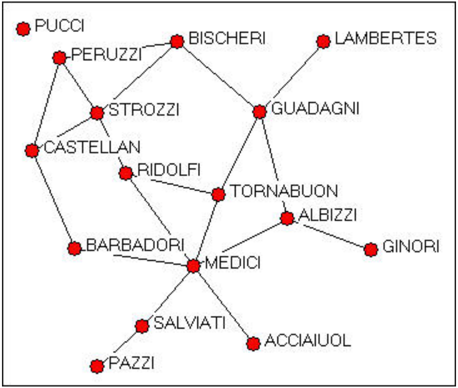
- Graph **center** is a set of of vertices with graph eccentricity equal to the graph radius $\epsilon(v) = r$ - set of central points
- Graph **periphery** is a set of vertices that have graph eccentricities equal to the graph diameter $\epsilon(v) = d$



from Eric Weisstein *MathWorld*

Centrality Measures

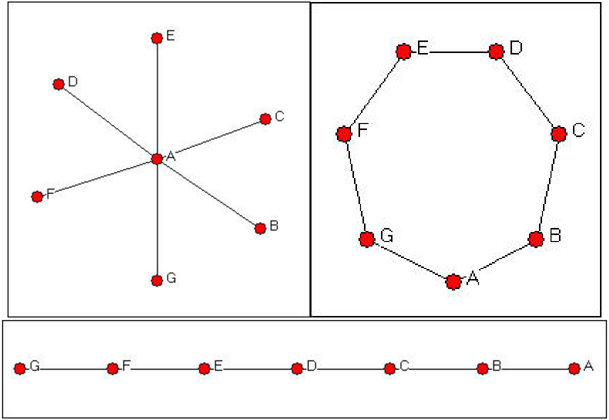
Sociology: determine the most "important" or "prominent" actors in the network based on actor location, involvement with other actors



Marriage alliances among leading Florentine families 15th century.

Padgett, 1993

Three graphs



Star graph Circle graph Line Graph

Degree centrality

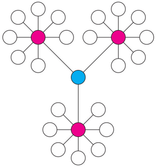
Degree centrality: number of nearest neighbours

$$C_D(i) = k(i) = \sum_j A_{ij} = \sum_j A_{ji}$$

Normalized degree centrality

$$C_D^*(i) = \frac{1}{n-1} C_D(i) = \frac{k(i)}{n-1}$$

High centrality degree -direct contact with many other actors



Closeness centrality

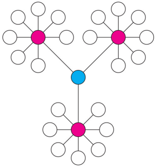
Closeness centrality: how close an actor to all the other actors in network

$$C_C(i) = \frac{1}{\sum_j d(i,j)}$$

Normalized closeness centrality

$$C_C^*(i) = (n-1)C_C(i) = \frac{n-1}{\sum_j d(i,j)}$$

High closeness centrality - short communication path to others, minimal number of steps to reach others



[*** Harmonic centrality $C_H(i) = \sum_j \frac{1}{d(i,j)}$ ***]

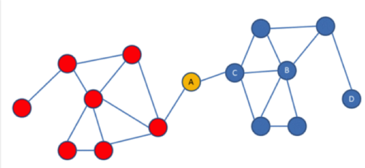
Alex Bavelas, 1948

Betweenness centrality: number of shortest paths going through the actor $\sigma_{st}(i)$

$$C_B(i) = \sum_{s \neq t \neq i} \frac{\sigma_{st}(i)}{\sigma_{st}}$$

Normalized betweenness centrality

$$C_B^*(i) = \frac{2}{(n-1)(n-2)} C_B(i) = \frac{2}{(n-1)(n-2)} \sum_{s \neq t \neq i} \frac{\sigma_{st}(i)}{\sigma_{st}}$$



High betweenness centrality - vertex lies on many shortest paths

Probability that a communication from s to t will go through i (geodesics)

Linton Freeman, 1977

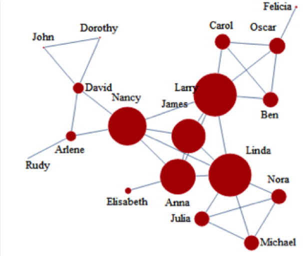
Eigenvector centrality

Importance of a node depends on the importance of its neighbors (recursive definition)

$$v_i \leftarrow \sum_j A_{ij} v_j$$

$$v_i = \frac{1}{\lambda} \sum_j A_{ij} v_j$$

$$\mathbf{Av} = \lambda \mathbf{v}$$

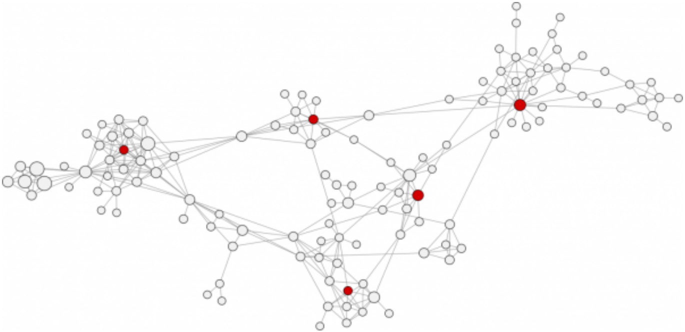


Select an eigenvector associated with largest eigenvalue $\lambda = \lambda_1$, $\mathbf{v} = \mathbf{v}_1$

Phillip Bonacich, 1972.

Centrality examples

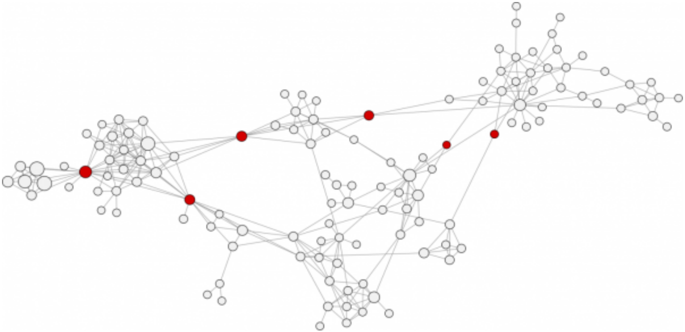
Closeness centrality



from www.activenetworks.net

Centrality examples

Betweenness centrality



from www.activenetworks.net

Centrality examples

Eigenvector centrality

from www.activenetworks.net

Katz status index

Weighted count of all paths coming to the node: the weight of path of length n is counted with attenuation factor β^n, $\beta < \frac{1}{\lambda_1}$

$$k_i = \beta \sum_j A_{ij} + \beta^2 \sum_j A_{ij}^2 + \beta^3 \sum_j A_{ij}^3 + \ldots$$

$$\mathbf{k} = (\beta \mathbf{A} + \beta^2 \mathbf{A}^2 + \beta^3 \mathbf{A}^3 + \ldots)\mathbf{e} = \sum_{n=1}^{\infty} (\beta^n \mathbf{A}^n)\mathbf{e} = (\sum_{n=0}^{\infty} (\beta \mathbf{A})^n - \mathbf{I})\mathbf{e}$$

$$\sum_{n=0}^{\infty} (\beta \mathbf{A})^n = (\mathbf{I} - \beta \mathbf{A})^{-1}$$

$$\mathbf{k} = ((\mathbf{I} - \beta \mathbf{A})^{-1} - \mathbf{I})\mathbf{e}$$

$$(\mathbf{I} - \beta \mathbf{A})\mathbf{k} = \beta \mathbf{A} \mathbf{e}$$

$$\mathbf{k} = \beta \mathbf{A} \mathbf{k} + \beta \mathbf{A} \mathbf{e}$$

$$\mathbf{k} = \beta (\mathbf{I} - \beta \mathbf{A})^{-1} \mathbf{A} \mathbf{e}$$

Leo Katz, 1953

Bonacich centrality

Two-parametric centrality measure $c(\alpha, \beta)$
β - radius of power, α - normalization parameter,
$\beta > 0$ - tied to more central (powerful) people
$\beta < 0$ - tied to less central (powerful) people
$\beta = 0$ - degree centrality

$$c_i(\alpha, \beta) = \sum_j (\alpha + \beta c_j) A_{ij}$$

$$\mathbf{c} = \alpha \mathbf{A} \mathbf{e} + \beta \mathbf{A} \mathbf{c}$$

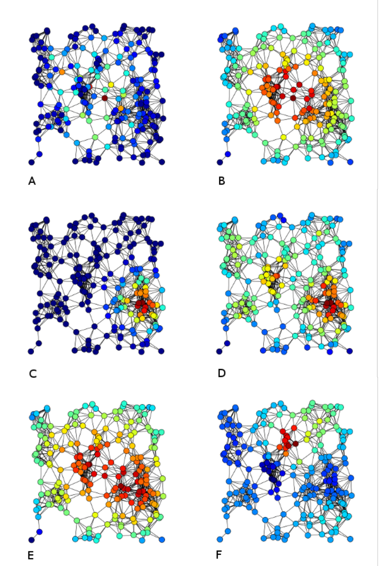
$$(\mathbf{I} - \beta \mathbf{A}) \mathbf{c} = \alpha \mathbf{A} \mathbf{e}$$

$$\mathbf{c} = \alpha (\mathbf{I} - \beta \mathbf{A})^{-1} \mathbf{A} \mathbf{e}$$

Normalizaton: $||\mathbf{c}||_2 = \sum c_i^2 = 1$

Phillip Bonacich, 1987

Centrality examples



from Wikipedia

- A) Betweenness centrality
- B) Closeness centrality
- C) Eigenvector centrality
- D) Degree centrality
- F) Harmonic centrality
- E) Katz centrality

Centralization

Centralization (network measure) - how central the most central node in the network in relation to all other nodes.

$$C_x = \frac{\sum_i^N [C_x(p_*) - C_x(p_i)]}{\max \sum_i^N [C_x(p_*) - C_x(p_i)]}$$

C_x - one of the centrality measures
p_* - node with the largest centrality value
max - is taken over all graphs with the same number of nodes (for degree, closeness and betweenness the most centralized structure is the star graph)

Linton Freeman, 1979

Metrics comparison

- **Pearson correlation** coefficient

$$r = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2}\sqrt{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}}$$

  Shows linear dependence between variables, $-1 \le r \le 1$
  (perfect when related by linear function)

- **Spearman rank correlation** coefficient (Sperman's rho):
  Convert raw scores to ranks - sort by score: $X_i \to x_i$, $Y_i \to y_i$

$$\rho = 1 - \frac{6\sum_{i=1}^{n}(x_i - y_i)^2}{n(n^2 - 1)}$$

  Shows strength of monotonic association
  (perfect for monotone increasing/decreasing relationship)

Ranking comparison

- The Kendall tau rank distance is a metric that counts the number of pairwise disagreements between two ranking lists
- Kendall rank correlation coefficient, commonly referred to as Kendall's tau coefficient

$$\tau = \frac{n_c - n_d}{n(n-1)/2}$$

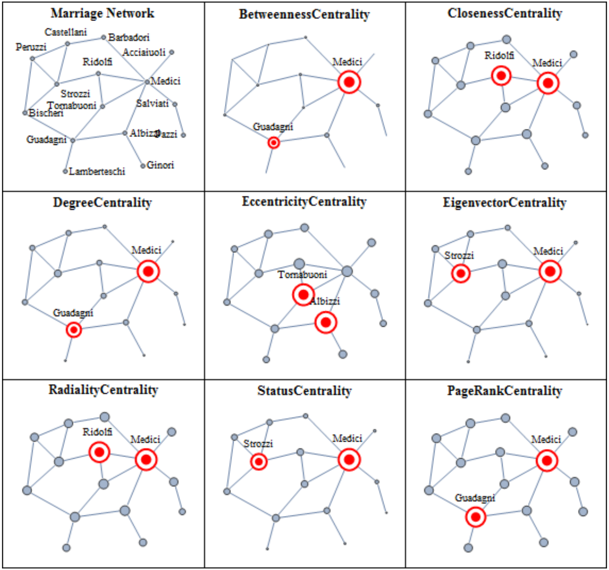
n_c - number of concordant pairs, n_d - number of discordant pairs

- $-1 \leq \tau \leq 1$, perfect agreement $\tau = 1$, reversed $\tau = -1$
- Example

Rank 1	A	B	C	D	E
Rank 2	C	D	A	B	E

$$\tau = \frac{6-4}{5(5-1)/2} = 0.2$$

Florentines families



References

- Centrality in Social Networks. Conceptual Clarification, Linton C. Freeman, Social Networks, 1, 215-239, 1979
- Power and Centrality: A Family of Measures, Phillip Bonacich, The American Journal of Sociology, Vol. 92, No. 5, 1170-1182, 1987
- A new status index derived from sociometric analysis, L. Katz, Psychometrika, 19, 39-43, 1953.
- Eigenvector-like measures of centrality for asymmetric relations, Phillip Bonacich, Paulette Lloyd, Social Networks 23, 191-201, 2001