Network structure

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Network Science

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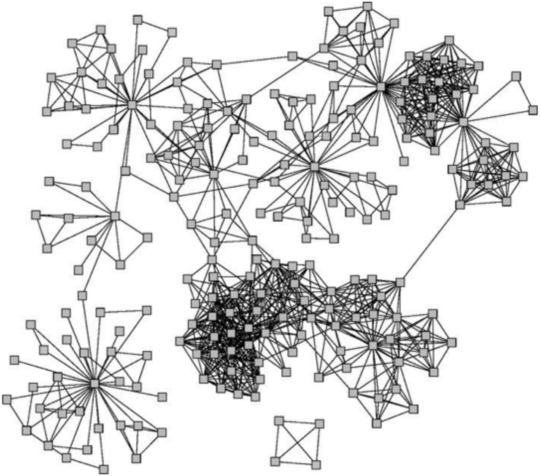


Typical network structure

Core-periphery structure of a network

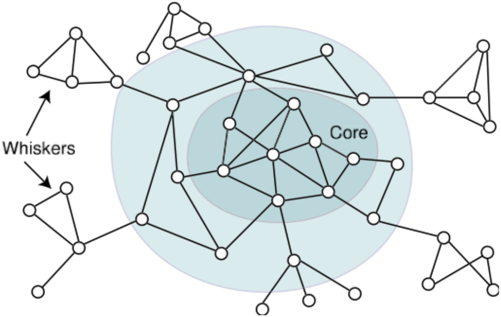
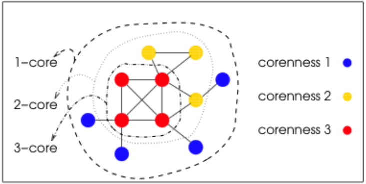


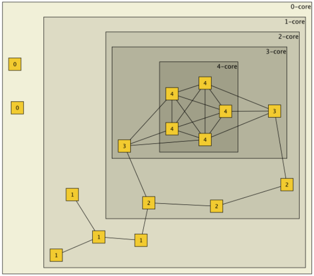
image from J. Leskovec, K. Lang, 2010

Graph cores

Definition

A *k-core* is the largest subgraph such that each vertex is connected to at least k others in subset





Every vertex in k-core has a degree $k_i \geq k$

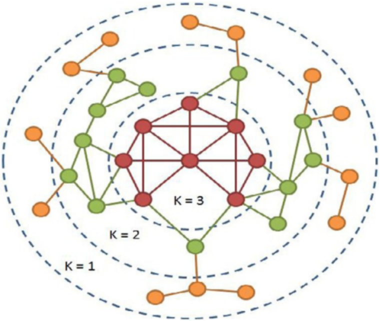
$(k+1)$-core is always subgraph of k-core

The core number of a vertex is the highest order of a core that contains this vertex

k-core decomposition

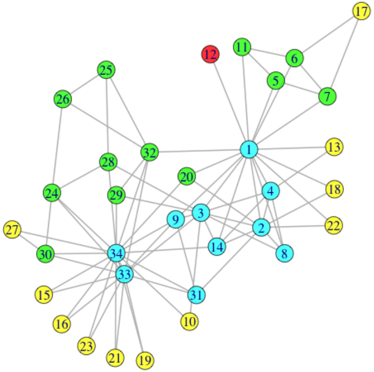
V. Batageli, M. Zaversnik, 2002

- If from a given graph G = (V, L) recursively delete all vertices, and lines incident with them, of degree less than k, the remaining graph is the k-core.

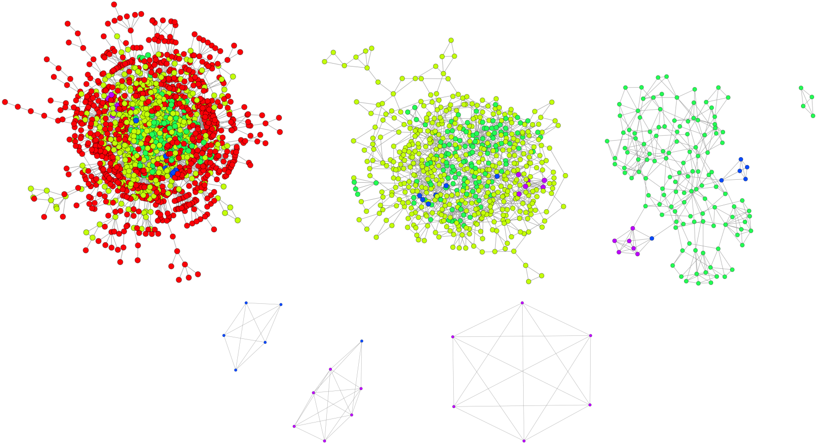


K-cores

Zachary karate club: 1,2,3,4 - cores



k-cores



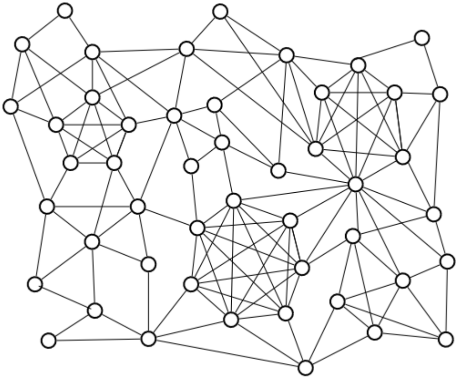
k-cores: 1:1458, 2:594, 3:142, 4:12, 5:6

k-shells: 1:864-red, 2:452-pale green, 3:130-green, 5:6-blue, 6:6-purple

Graph cliques

Definition

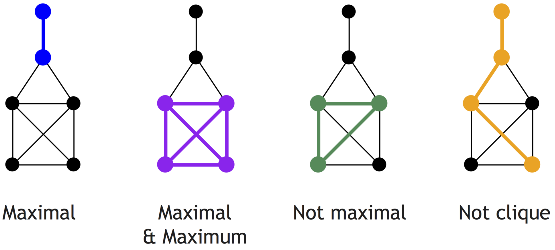
A *clique* is a complete (fully connected) subgraph, i.e. a set of vertices where each pair of vertices is connected.



Cliques can overlap

Graph cliques

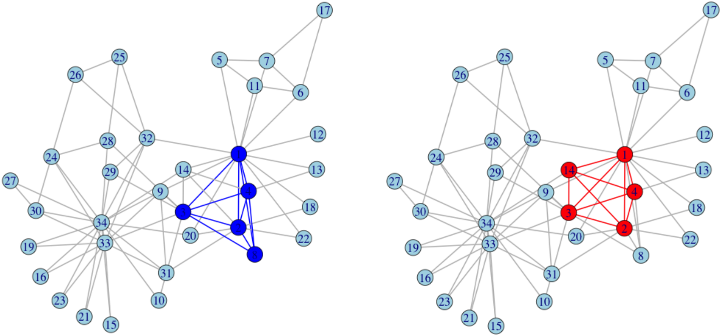
- A **maximal clique** is a clique that cannot be extended by including one more adjacent vertex (not included in larger one)
- A **maximum clique** is a clique of the largest possible size in a given graph



- Graph clique number is the size of the maximum clique

image from D. Eppstein

Maximum cliques



Maximal cliques:

Clique size:	2	3	4	5
Number of cliques:	11	21	2	2

Zachary, 1977

Graph cliques

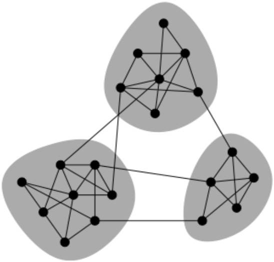
Computational issues:

- Finding click of fixed given size k - $O(n^k k^2)$
- Finding maximum clique $O(3^{n/3})$
- But in sparse graphs...

Network communities

Definition

Network communities are groups of vertices such that vertices inside the group connected with many more edges than between groups.



- Community detection is an assignment of vertices to communities.
- Will consider non-overlapping communities
- Graph partitioning problem

Network communities

What makes a community (cohesive subgroup):

- Mutuality of ties. Almost everyone in the group has ties (edges) to one another
- Compactness. Closeness or reachability of group members in small number of steps, not necessarily adjacency
- Density of edges. High frequency of ties within the group
- Separation. Higher frequency of ties among group members compared to non-members

Wasserman and Faust

Community density

- Graph $G(V, E)$, $n = |V|$, $m = |E|$
- Community - set of nodes S
  n_s-number of nodes in S, m_s - number of edges in S
- Graph density
$$\rho = \frac{m}{n(n-1)/2}$$
- community internal density
$$\delta_{int}(C) = \frac{m_s}{n_s(n_s-1)/2}$$
- external edges density
$$\delta_{ext}(C) = \frac{m_{ext}}{n_c(n-n_c)}$$
- community (cluster): $\delta_{int} > \rho$, $\delta_{ext} < \rho$

Modularity

- Compare fraction of edges within the cluster to expected fraction in random graph with identical degree sequence

$$Q = \frac{1}{4}(m_s - E(m_s))$$

- Modularity score

$$Q = \frac{1}{2m}\sum_{ij}\left(A_{ij} - \frac{k_i k_j}{2m}\right)\delta(c_i, c_j), = \sum_u (e_{uu} - a_u^2)$$

  e_{uu} - fraction of edges within community u
  $a_u = \sum_u e_{uv}$ - fraction of ends of edges attached to nodes in u

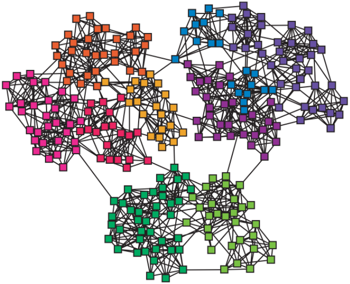
- The higher the modularity score - the better are communities
- Modularity score range $Q \in [-1/2, 1)$, single community $Q = 0$

Heuristic approach

Focus on edges that connect communities.

Edge betweenness -number of shortest paths $\sigma_{st}(e)$ going through edge e

$$C_B(e) = \sum_{s \neq t} \frac{\sigma_{st}(e)}{\sigma_{st}}$$



Construct communities by progressively removing edges

Edge betweenness

Newman-Girvan, 2004

Algorithm: Edge Betweenness

Input: graph G(V,E)

Output: Dendrogram

repeat

For all $e \in E$ compute edge betweenness $C_B(e)$;

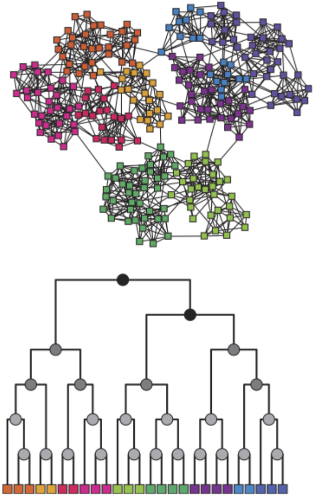
remove edge e_i with largest $C_B(e_i)$;

until *edges left*;

If bi-partition, then stop when graph splits in two components (check for connectedness)

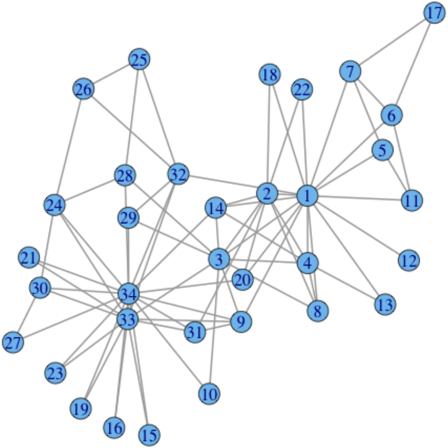
Edge betweenness

Hierarchical algorithm, dendrogram



Edge betweenness

Zachary karate club



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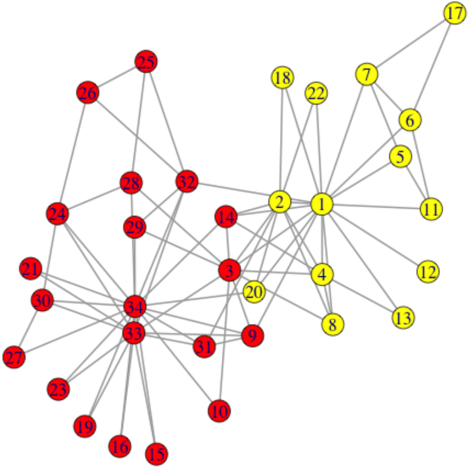

Edge betweenness

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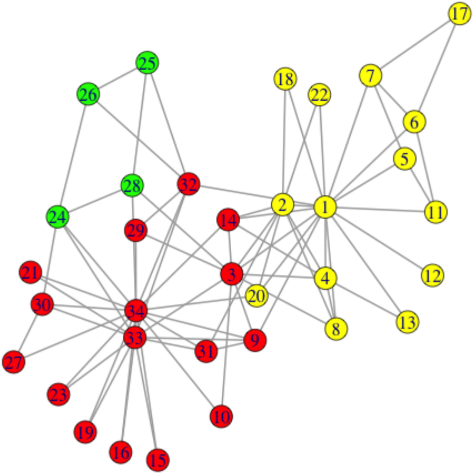
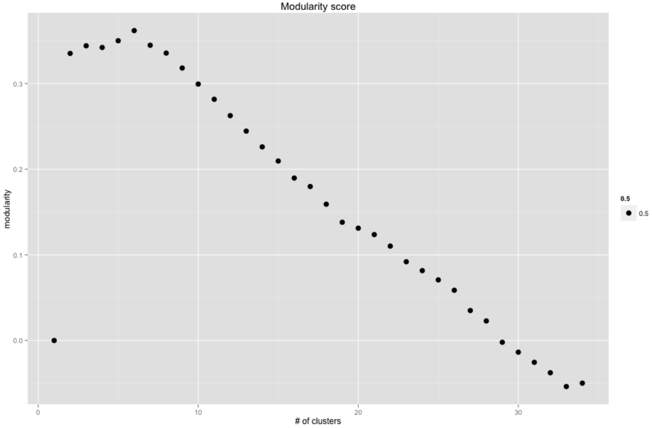
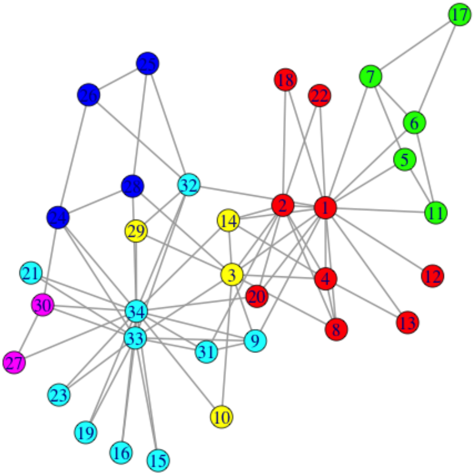


Edge betweenness



Edge betweenness

best: clusters = 6, modularity = 0.345



Edge betweenness

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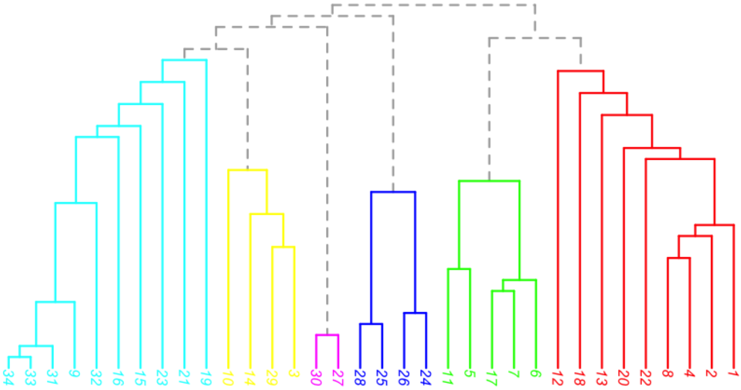

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