NATIONAL RESEARCH UNIVERSITY

Mathematical models of networks

Social Network Analysis. MAGoLEGO course.
Lecture 3

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Network models

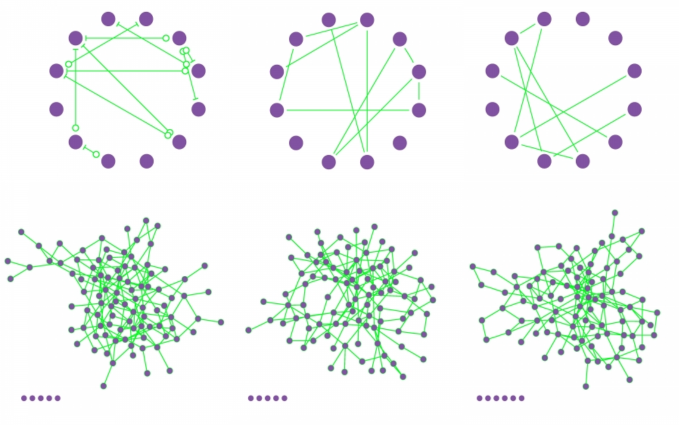
network features:

- Power-law (heavy-tailed) degree distribution
- Small average distance (graph diameter)
- Large clustering coefficient (transitivity)
- Giant connected component

Generative models:

- Random graph model (Erdos & Renyi, 1959)
- Preferential attachement model (Barabasi & Albert, 1999)
- Small world model (Watts & Strogatz, 1998)

Random graph model



top: $n = 12, p = 1/6$ bottom: $n = 100, p = 0.03$

Random graph model

Graph $G\{E, V\}$, nodes $n = |V|$, edges $m = |E|$
Erdos and Renyi, 1959.

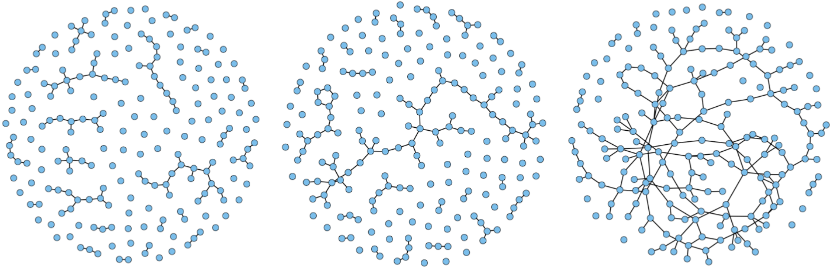
- $G_{n,p}$ - each pair out of $N = \frac{n(n-1)}{2}$ is connected with probability p,
  number of edges m - random number

$$\langle m \rangle = p\frac{n(n-1)}{2}$$

$$\langle k \rangle = \frac{1}{n}\sum_i k_i = \frac{2\langle m \rangle}{n} = p\left(n-1\right) \approx pn$$

$$\rho = \frac{\langle m \rangle}{n(n-1)/2} = p$$

Random graph model



$p < p_c$ $p = p_c$ $p > p_c$

Structural changes happens when increasing p

igraph:erdos.renyi.game()

Phase transition

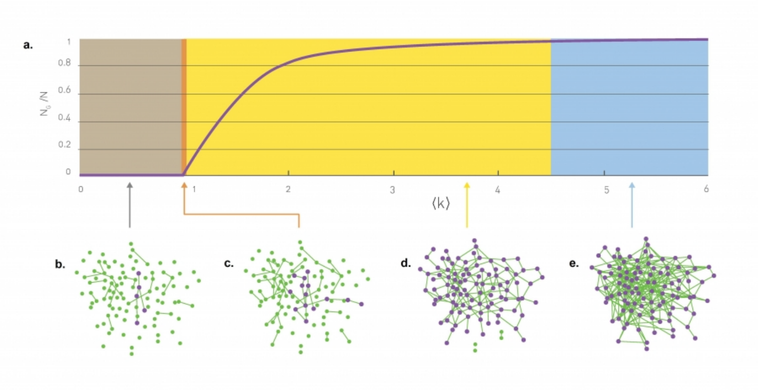
Consider $G_{n,p}$ as a function of p

- $p = 0$, empty graph
- $p = 1$, complete (full) graph
- There are exist critical p_c:
  - when $p < p_c$, $(\langle k \rangle < 1)$ there is no components with more than $O(\ln n)$ nodes, largest component is a tree
  - when $p = p_c$, $(\langle k \rangle = 1)$ the largest component has $O(n^{2/3})$ nodes
  - when $p > p_c$, $(\langle k \rangle > 1)$ gigantic component has all $O(n)$ nodes

Critical value: $\langle k \rangle = p_c n = 1$- on average one neighbor for a node

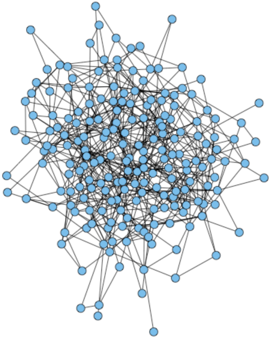
Phase transition

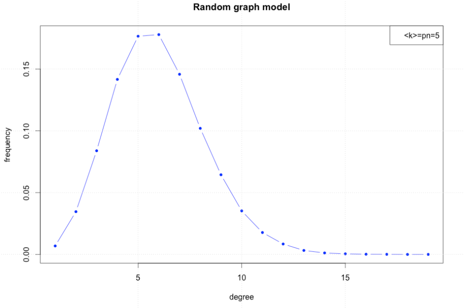
The size of largest connected component



Critical value: $\langle k \rangle = p_c n = 1$- on average one neighbor for a node

Random graph



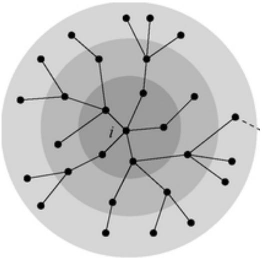


Node degree distribution (Poisson distribution):

$$P(k_i = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad \lambda = pn = \langle k \rangle$$

Graph diameter

- $G(n, p)$ is locally tree-like (GCC) (no loops; low clustering coefficient)



- on average, the number of nodes d steps away from a node $\langle k \rangle^d$
- in GCC, around p_c , $\langle k \rangle^d \sim n$,

$$d \sim \frac{\ln n}{\ln \langle k \rangle}$$

Clustering coefficient

- Clustering coefficient

$$C(k) = \frac{\text{\#of links between NN}}{\text{\#max number of links NN}} = \frac{pk(k-1)/2}{k(k-1)/2} = p$$

$$C = p = \frac{\langle k \rangle}{n}$$

- when $n \to \infty, \;\; C \to 0$

Random graph model

- Node degree distribution function - Poisson:

$$P(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \ \ \lambda = pn = \langle k \rangle$$

- Average path length:

$$\langle L \rangle \sim \log(N) / \log\langle k \rangle$$

- Clustering coefficient:

$$C = \frac{\langle k \rangle}{n}$$

Preferential attachment model

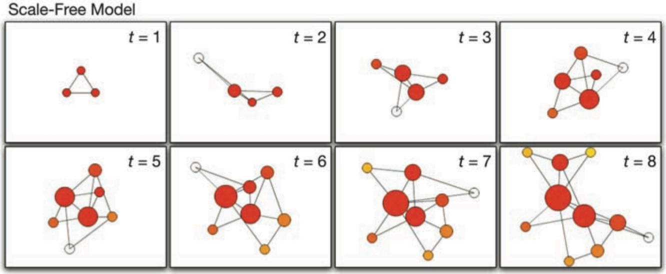
Barabasi and Albert, 1999
Dynamical growth model

- $t = 0$, n_0 nodes
- growth: on every step add a node with m_0 edges ($m_0 \leq n_0$), $k_i(i) = m_0$
- Preferential attachment: probability of linking to existing node is proportional to the node degree k_i

$$\Pi(k_i) = \frac{k_i}{\sum_i k_i} = \frac{k_i}{2m_0 t}$$

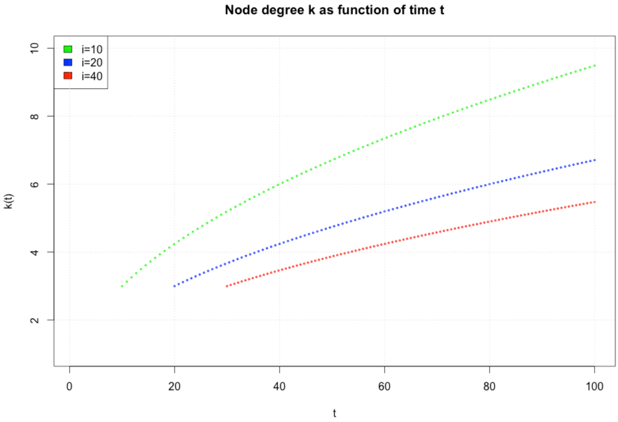
after t steps: $n_0 + t$ nodes, $m_0 t$ edges

Preferential attachment model



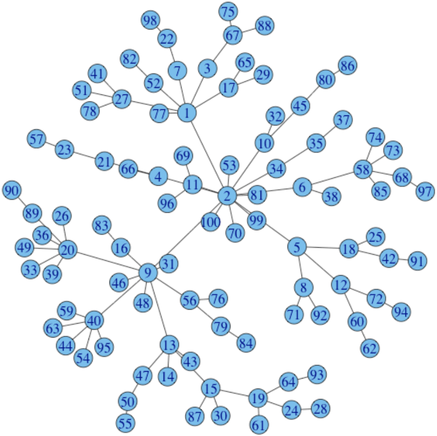
Barabasi, 1999

Preferential attachement

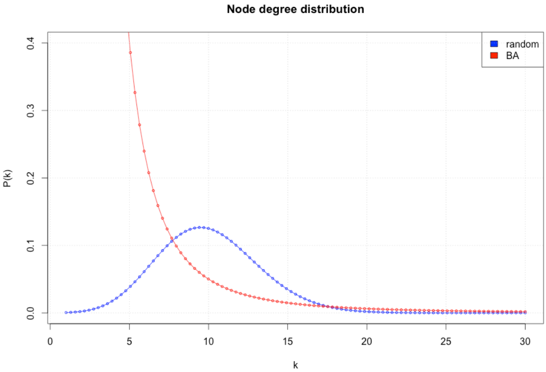


$$k_i(t) = m_0 \left(\frac{t}{i}\right)^{1/2}$$

Preferential attachement



Preferential attachement

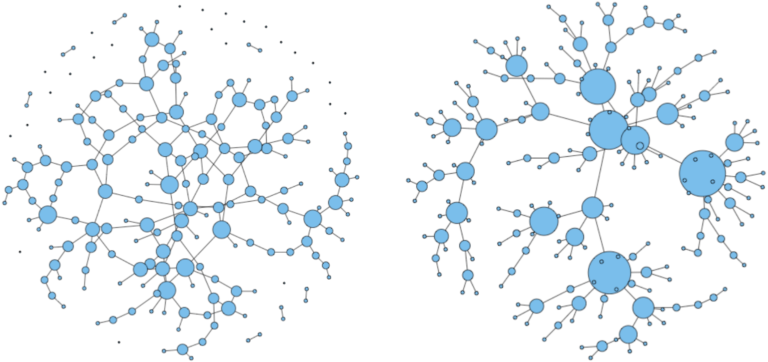


Node degree distribution:

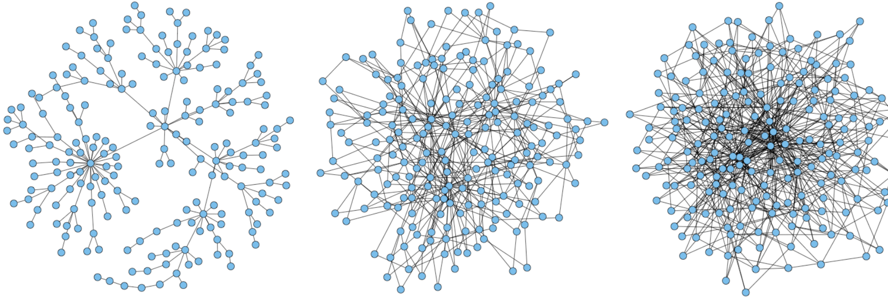
$$P(k_i = k) = \frac{2m_0^2}{k^3}$$

Preferential attachement

Preferential attachment vs random graph



Preferential attachment model



$m_0 = 1$ $m_0 = 2$ $m_0 = 3$

igraph: barabasi.game()

Preferential attachment model

- Node degree distribution - power law):

$$P(k) = \frac{2m_0^2}{k^3}$$

- Average path length :

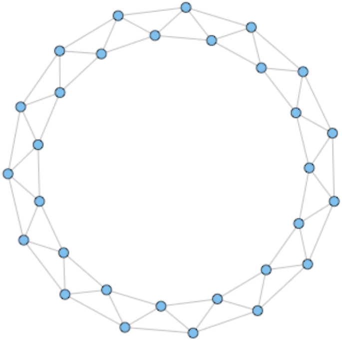
$$\langle L \rangle \sim \log(N) / \log(\log(N))$$

- Clustering coefficient (numerical result):

$$C \sim N^{-0.75}$$

Small world

Motivation: keep high clustering, get small diameter



Clustering coefficient $C = 1/2$
Graph diameter $d = 8$

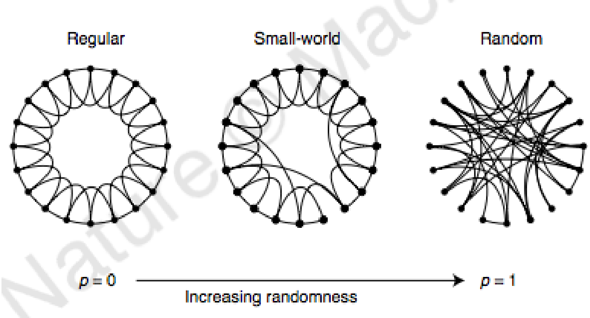
Small world

Watts and Strogatz, 1998

Single parameter model, interpolation between regular lattice and random graph

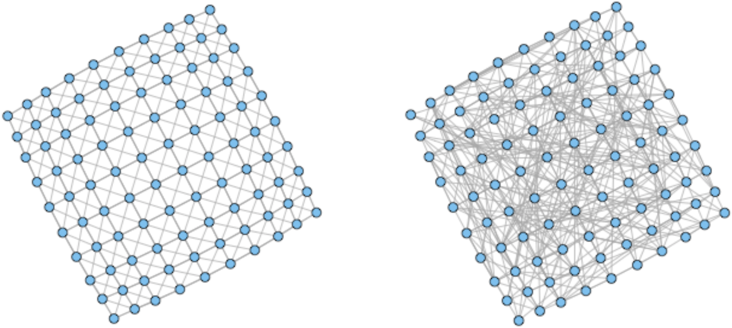
- start with regular lattice with n nodes, k edges per vertex (node degree), $k << n$
- randomly connect with other nodes with probability p, forms $pnk/2$ "long distance" connections from total of $nk/2$ edges
- $p = 0$ regular lattice, $p = 1$ random graph

Small world



Watts, 1998

Small world model



20% rewiring:

ave. path length = 3.58 → ave. path length = 2.32

clust. coeff = 0.49 → clust. coeff = 0.19

igraph:watts.strogatz.game()

Small world model

- Node degree distribution function - Poisson like (numerical result)
- Average path length (analytical result) :

$$\langle L \rangle \sim \log(N)$$

- Clustering coefficient

$$C = const$$

Model comparison

	Random	BA model	WS model	Empirical networks
P(k)	$\frac{\lambda^k e^{-\lambda}}{k!}$	k^{-3}	poisson like	power law
C	$\langle k \rangle / N$	$N^{-0.75}$	const	large
$\langle L \rangle$	$\frac{\log(N)}{\log(\langle k \rangle)}$	$\frac{\log(N)}{\log \log(N)}$	$\log(N)$	small

References

- On random graphs I, P. Erdos and A. Renyi, Publicationes Mathematicae 6, 290–297 (1959).
- On the evolution of random graphs, P. Erdos and A. Renyi, Publicaton of the Mathematical Institute of the Hungarian Academy of Sciences, 17-61 (1960)
- Collective dynamics of small-world networks. Duncan J. Watts and Steven H. Strogatz. Nature 393 (6684): 440-442, 1998
- Emergence of Scaling in Random Networks, A.L. Barabasi and R. Albert, Science 286, 509-512, 1999