Dirac fermions with random velocity

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Abstract

Dirac fermions on a two-dimensional lattice are considered. Hopping integrals are chosen to be random and describe random velocity $v(x, y)$ slowly-varying along the plane. We demonstrate that low-lying excitations of the system correspond to the "edge-states" propagating along the lines $v(x, y) = 0$. The dispersion law of these excitations is found to be $E(k) \propto 1/|\ln(kd)|$, where d is the lattice constant. Similarly to the conventional edge states in the quantum Hall systems, there is no back-scattering for electrons propagating along the zero-velocity lines. But in contrast to the motion in magnetic field the transverse size of the wave function is on the order of k^{-1} and goes to infinity as E approaches zero. Applications of the results to the realistic two-dimensional systems with random spin-orbit coupling are discussed.

PACS numbers: 72.15.Rn, 73.40.Hm, 78.20.Ek

Dirac fermions have recently attracted a lot of attention in connection with the integer quantum Hall effect (QHE) and high temperature superconductivity (HTSC).

In the context of QHE spinless Dirac fermions allow to model the "edge" states, which describe unidirectional propagation of electrons in a magnetic field. The analogy arises from the situation when the mass of a fermion (originating from the Zeeman effect) varies along some direction and passes through zero. Then the eigenstates of Dirac's fermions represent the so called "snake states" propagating along the line $M = 0$[1,2].

One of the proposed mechanisms for HTSC in oxide superconductors is a d-wave pairing. In the absence of impurities the Hamiltonian of the tight-binding model on the square lattice can be easily diagonalized in the momentum representation and yields an energy spectrum $E(\mathbf{k}) = \pm 2t\sqrt{\cos^2 k_x + \cos^2 k_y}$. This model produces doubled Dirac fermions located at the points $\mathbf{k} = (\pi/2, \pm\pi/2)$[3]. The absence of a real gap within this model is considered to be a crucial distinguishing feature of the d-wave superconductivity. However, if disorder is present, it can result in existence of localized states around zero energy and thus mimic the activation behavior. This would make the d-wave superconductors experimentally indistinguishable from the BCS[4]. This, hence, makes it very important to investigate the effect of disorder on Dirac fermions[5].

The first study of low energy region of Dirac fermions with disorder was carried out by P.A. Lee in 1993[4]. It was followed by the numerical study by Hatsugai and Lee[3]. The types of disorder considered in those papers were either "white noise" or short-range impurity potential. The results of[4,3] support the proposal of localization effects in a gapless d-wave superconductor.

Historically 2D massless Dirac fermions appeared in the paper M. P. A. Fisher and E. Fradkin[6], who considered spinless tight-binding model with the nearest neighbors hopping and a half flux quantum ("π flux") per plaquette. The authors used the disorder in the form of random potential and came to the conclusion that it would lead to a finite density of states at zero energy (this conclusion is still a subject of a controversy[5,7,8]). However, at that time the model[6] was considered primarily of theoretical interest, since its direct realization

required very high magnetic fields $\sim 10^9 G$.

In connection with QHE the conventional type of disorder considered is a random magnetic field together with a random impurity potential[1]. With this type of disorder included, Dirac fermions describe the transition between different quantum Hall states and are relevant to the problem of impurity driven transition between quantum Hall plateaus[9]. The current focus of interest is multifractal behavior of Dirac fermion's eigenfunctions[10–13].

Another plausible realization of Dirac fermions is a 2D electron gas in semiconductor heterostructures. Consider an electron with a spin moving in the plane which is a boundary between two layers, so that there exists a gradient of the potential in the direction perpendicular to the plane. The origin of the gradient is either an asymmetry of the confinement potential[14], or "two-dimensionalizing" of the bulk periodic potential[15]. In this particular realization the disorder results from the inhomogeneity of ∇U.

The conventional way to investigate the role of disorder is to choose it in the form of a "white noise." This choice offers obvious advantages in terms of numerical computations. However, the qualitative picture in this approach is missing. It is well known that in order to get an insight into the structure of electronic states, one should resort to the case of a smooth disorder[17]. The above argument was a motivation for our research.

To be specific, consider the Rashba-Bychkov Hamiltonian[14]

$$\hat{H} = \alpha[\mathbf{k} \times \sigma]\mathbf{n} = \begin{pmatrix} 0 & \alpha(k_y + ik_x) \\ \alpha(k_y - ik_x) & 0 \end{pmatrix} \tag{1}$$

where parameter α has a meaning of velocity and varies smoothly with coordinates. Here $\mathbf{k} = (k_x, k_y)$ is in-plane momentum, σ are the Pauli matrices and $\mathbf{n}$ is a normal to the plane. From the first glance, we might assume that locally the spectrum has the form $E(k) = \pm\alpha\,|k|$ with a *local* value of $\alpha = \alpha(x, y)$. Then the density of states for the entire system can be readily obtained by averaging over all possible configurations of α

$$\langle \rho(E) \rangle = \frac{1}{\pi} \int \rho_0(E, \alpha) f(\alpha) d\alpha = \frac{1}{\pi} \int \frac{|E|}{\alpha^2} f(\alpha) d\alpha, \tag{2}$$

where $\rho(E, \alpha) = |E|/\pi\alpha^2$ is the density of states for clean Dirac fermions and $f(\alpha)$ is the

distribution function of the velocity α. As one can see the density of states computed in this way diverges in the regions where $\alpha \approx 0$ for *all* values of E. From here we conclude that the regions where α changes sign are very important and require further investigation. This investigation is carried out in the next section.

I. DIRAC FERMIONS WITH VARYING VELOCITY

In the vicinity of the contour line $\alpha(x,y) = 0$, α changes in the direction perpendicular to the contour and its spatial dependence can be linearized. We specify α in the form $\alpha(x,y) = \gamma x$, so it passes through zero at $x = 0$. The Hamiltonian for the varying velocity case can be obtained from (1) by adding a commutator of $\frac{1}{2}[\alpha, \hat{\mathbf{k}}]$ to the Hamiltonian. This commutator is nonzero, since now α depends on x. Adding the commutator to (1) insures the hermicity of the Hamiltonian. Then it takes the form

$$\hat{H} = \begin{pmatrix} 0 & \frac{1}{2}\gamma + \gamma x \hat{k}_y + \gamma x \frac{\partial}{\partial x} \\ -\frac{1}{2}\gamma + \gamma x \hat{k}_y - \gamma x \frac{\partial}{\partial x} & 0 \end{pmatrix}. \tag{3}$$

Since Hamiltonian does not depend on y, the wave vector k_y is a good quantum number, and we can search for the solution in the form

$$\Psi(x,y) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix} \exp(ik_y y). \tag{4}$$

Upon substitution of (4) into (3) we get the following system of equations

$$\begin{aligned} E\Psi_1 &= \frac{1}{2}\gamma\Psi_2 + \gamma x k_y \Psi_2 + \gamma x \frac{d\Psi_2}{dx} \\ E\Psi_2 &= -\frac{1}{2}\gamma\Psi_1 + \gamma x k_y \Psi_1 - \gamma x \frac{d\Psi_1}{dx}. \end{aligned} \tag{5}$$

Expressing one function through the other, the system (5) can be reduced to a single equation:

$$\frac{d^2\Psi_2}{dz^2} + \frac{2}{z}\frac{d\Psi_2}{dz} + \left(\frac{1}{z} + \frac{\varepsilon^2 + 1/4}{z^2} - 1\right)\Psi_2 = 0, \tag{6}$$

which has a form of a radial Schroedinger equation with an effective potential

$$U(z,\varepsilon) = -\left(\frac{1}{z} + \frac{\varepsilon^2 + 1/4}{z^2}\right). \tag{7}$$

Here $z = k_y x$ and $\varepsilon = E/\gamma$ are the dimensionless coordinate and energy respectively. At small distances $U(z)$ is an *attractive* potential, which means that an electron is attracted to the $x = 0$ boundary line.

Note that Eq. (6) formally coincides with the Schroedinger equation for an electron in hydrogen atom with "Coulomb " $1/\rho$ and "centrifugal" $l(l+1)/\rho^2$ terms. However in our case the corresponding "angular momentum" $l = 1/2 \pm i\varepsilon$ appears to be complex. Following the procedure described in[16] we can estimate the asymptotics of the solution for small values of z in the following way.

Let us choose some point z_0 within a small vicinity of the origin. Within the region $z < z_0$ we substitute $(\varepsilon^2 + \frac{1}{4})/z^2$ by $(\varepsilon^2 + \frac{1}{4})/z_0^2$. Then we can find the wave function inside this region and outside it. Then we can match the wave functions with their derivatives at the boundary $z = z_0$. For $z < z_0$ Eq. (6) takes the form

$$\frac{d^2\Psi_2}{dz^2} + \frac{2}{z}\frac{d\Psi_2}{dz} + \frac{\varepsilon^2 + \frac{1}{4}}{z_0^2}\Psi_2 = 0, \tag{8}$$

yielding the nondiverging solution in the origin

$$\Psi_2 = C\frac{\sin(\frac{z}{z_0}\sqrt{(\varepsilon^2 + \frac{1}{4})})}{z}. \tag{9}$$

For $z > z_0$ we look for the solution of Eq. (6) in the form $\Psi_2 = z^s$. This leads to the following quadratic equation for s

$$s(s+1) + \varepsilon^2 + \frac{1}{4} = 0, \tag{10}$$

which has two complex-conjugate roots:

$$\begin{aligned} s_1 &= -\frac{1}{2} + i\,|\varepsilon| \\ s_2 &= -\frac{1}{2} - i\,|\varepsilon|\,. \end{aligned} \tag{11}$$

Thus for Ψ_2 we have

$$\Psi_2 = Az^{s_1} + Bz^{s_2}. \tag{12}$$

Equating logarithmic derivatives at the boundary $z = z_0$ we solve for the ratio

$$\frac{B}{A} = const \times z_0^{s_1 - s_2}. \tag{13}$$

The real solution for small values of z will be

$$\Psi_2(z) = \frac{const}{\sqrt{z}} \cos(|\varepsilon| \ln(\frac{z}{z_0}) + const). \tag{14}$$

Substituting Ψ_2 back into the system (14) we find Ψ_1

$$\Psi_1(z) = \frac{const}{\sqrt{z}} \sin(|\varepsilon| \ln(\frac{z}{z_0}) + const). \tag{15}$$

The choice of the constant phase in Ψ_1 and Ψ_2 is dictated by the condition that Ψ has to be zero at $z \to \infty$. In principle, the solutions of Eq. (6) can be expressed through hypergeometrical function and have the asymptotic behavior $\sim e^z$ and e^{-z} at large z. The phase constant should be chosen in such a way that the coefficient in front of e^z vanishes. We can avoid dealing with hypergeometric function by considering the limit as $\varepsilon \to 0$. In this limit the system (5) decouples and has an obvious solution

$$\begin{aligned} \Psi_1 &= \frac{1}{\sqrt{|x|}} \exp(k_y x), x < 0 \\ \Psi_2 &= \frac{1}{\sqrt{x}} \exp(-k_y x), x > 0. \end{aligned} \tag{16}$$

Then in order for (14) and (15) to satisfy the limiting case (16), the phase constant should vanish at $\varepsilon \to 0$. This means, that in both (14) and (15) the phase constant can be combined with z_0.

This solution (14) is plotted in Fig. 1. From this solution we can immediately see that the wave function does not have a $z \to 0$ limit: oscillations increase in amplitude and frequency as z decreases. This behavior suggests that we need to introduce some cut-off parameter in the problem.

Let us denote the cut-off distance with d. Quantization of levels in a potential well corresponds to the change of argument of the cos in Eq. (14) by $\sim \pi$ between turning points. In our case the turning points are $z \sim 1$ (external) and $z \sim k_y d$ (internal). Thus, Eq. (14) allows us to conjecture the form of the dispersion law $E(k_y)$ for the wave propagating along the boundary.

$$\frac{|E|}{\gamma} = |\varepsilon| = \frac{c_1}{\ln(c_2/kd)}, \tag{17}$$

where c_1 and c_2 are constants of the order of 1. The spectrum (17) is depicted in Fig. 2. The energy goes to zero at small k_y, and has an infinite derivative at the origin.

II. NUMERICAL RESULTS

The natural way to introduce a cut-off parameter in the problem is to reformulate it on the lattice. Then the discretized Schroedinger equation on the lattice with spacing d takes the form:

$$\begin{aligned} Ea_{n,m} &= c_n b_{n-1,m} + c_{n+1} b_{n+1,m} + \frac{i}{2}(c_n + c_{n+1})(b_{n,m+1} + b_{n,m-1}) \\ Eb_{n,m} &= c_n a_{n-1,m} + c_{n+1} a_{n+1,m} - \frac{i}{2}(c_n + c_{n+1})(a_{n,m+1} + a_{n,m-1}). \end{aligned} \tag{18}$$

with

$$a_n = \Psi_1(nd), b_n = \Psi_2(nd), c_n = \alpha(nd). \tag{19}$$

Following the above consideration we look for the solution in the form:

$$\Psi_{m,n} = \begin{pmatrix} A_n \\ B_n \end{pmatrix} i^{(m+n)} \exp(ikmd). \tag{20}$$

After substituting (20) into (18) we arrive at the following matrix equation

$$E \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 & Q_1 \\ Q_2 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \tag{21}$$

which can be easily solved numerically. Here, A and B are column vectors, and Q_1 and Q_2 are *tridiagonal* matrices.

$$\begin{aligned}(Q_1)_{n,n-1} &= -(Q_2)_{n,n-1} = -c_n \\ (Q_1)_{n,n} &= (Q_2)_{n,n} = (c_n + c_{n+1})\sin(kd) \\ (Q_1)_{n,n+1} &= -(Q_2)_{n,n+1} = c_{n+1}.\end{aligned} \tag{22}$$

Note that this system decomposes into two subsystems - one for vector A and one for vector B. Then we have $Q_1Q_2A = E^2A$ and $Q_2Q_1B = E^2B$. For each eigenvalue we have a trivial double degeneracy corresponding to electron and "hole" states, with energies $\pm E$.

To solve the system numerically we have to specify c_n. First we consider the above case $\alpha = \gamma x$, thus choosing $c_n = \gamma nd$. It is easy to see, that for this particular case the equations for $n > 0$ and $n < 0$ become independent, indicating that the solutions are localized either to the left or to the right from the boundary line (this is true for any case when boundary line goes through the lattice site).

In Fig. 3 we show these states. The three lowest branches of the energy spectrum for eigenstates of (21) are plotted in Fig. 4.To test our conjecture (17) we plot $1/|E(k_y)|$ as a function of $\ln(k_y)$. Eq. (17) provides a perfect fit for the numerical results: we observe that all branches of the spectrum became straight lines. Surprisingly, the linear dependence holds for kd up to 1, though for the estimate of the dispersion law (17) we used a much stronger assumption of $kd \ll 1$. Using the slope of the lines we can find the corresponding coefficients in (17)

$$\begin{aligned}c_1 = 1.5883, c_2 = 4.2879 \\ c_1 = 4.7649, c_2 = 4.2879 \\ c_1 = 7.9492, c_2 = 4.2879.\end{aligned} \tag{23}$$

We also note that within the accuracy of our computations c_2 is a constant.

Now, since we know the spectrum, we can calculate the density of states. In the region where α does not go over zero, we still can replace α by a constant local value and get

$$\rho(E) = \frac{|E|}{\pi\alpha^2}. \tag{24}$$

However, in the vicinity of the boundary line, where α changes sign, we should use the new dispersion law (17). Then, for our model with $c_n = \gamma nd$ we obtain:

$$\rho(E) = \sum_{k_y} \delta(E - E(k_y)) = \frac{\gamma c_1 c_2}{\pi E^2} \exp(-\frac{\gamma c_1}{|E|}). \tag{25}$$

Eq. (25) describes a one-dimensional density of boundary states, which vanishes in a singular fashion when $E \to 0$.

Next we address the following question. Will the boundary states we found persist for saturating $\alpha(x)$, as the realistic dependence $\alpha(x)$ should eventually saturate with increasing distance from the boundary? To study this question we choose $c_n = \gamma n_0 d \tanh(\frac{n}{n_0})$, so that $x_0 = nd$. It is obvious that the system (21) has a continuous spectrum, similar to that of $\alpha(x) = \alpha(\infty) = const$. We show the numerical results in Fig. 5. As can be seen, the discrete spectrum *does survive* the saturating behavior of $\alpha(x)$. The uniqueness of the situation presented in Fig. 5 is that the discrete spectrum coexists with continuum for the same values of energy. Discrete states *do not decay* into continuum as they have a much higher momentum k_y than allowed for continuum state $k_{y,max} = E/\alpha_\infty = E/\gamma n_0 d$. The separation line between the continuum and discrete parts of the spectrum is $E_c = \gamma n_0 k_y d$. Compare to the case of $c_n = \gamma n$ the spectrum "softens" at small k_y. Instead of the singular behavior $E \sim \ln^{-1}(\frac{1}{k_y})$ we have an acoustic like spectrum at small k_y, with a derivative restricted by the slope of separation line γn_0.

The physical picture of this state is the following: the continuum spectrum is formed by the plane waves propagating in all directions. Discrete corresponds to the boundary states that are localized within a region around the line $\alpha = 0$ and can propagate only along this line.

III. ABSENCE OF BACK-SCATTERING

The most remarkable property of the boundary states is the absence of back-scattering. Consider a short range impurity located close to the boundary $\alpha(x) = 0$. Usually, as, say, for waves propagating within a quantum wire, the impurity will cause a finite reflection of the wave with momentum k_y into the wave with $-k_y$. This conventional wisdom does not apply to the boundary states of Dirac fermions described above. The propagating boundary state will not "notice" the impurity and will not suffer any back-scattering. This happens because the matrix element for reflection is identically zero. It follows from the general property that for arbitrary $\alpha(x)$, if

$$\Psi_{k_y}(x,y) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix} \exp(ik_y y). \tag{26}$$

is a solution of the system, then

$$\Psi_{-k_y}(x,y) = \begin{pmatrix} \Psi_2(x) \\ -\Psi_1(x) \end{pmatrix} \exp(-ik_y y). \tag{27}$$

is also a solution, so that Ψ_{k_y} and Ψ_{-k_y} are orthogonal to each other at *any point*.

IV. TWO-BOUNDARY STATES

Since the size of the wave functions is $\sim 1/k$, it drastically increases with decreasing E (as $\exp(\frac{\gamma}{|E|})$). This means, than at low enough energies (for $k \ll 1/n_0 d$) the wave functions of neighboring boundary states start to overlap and form pairs. The next step towards understanding the global structure of boundary states of the Dirac fermions in a random system is to consider a pair of boundaries. We may expect the existence of states belonging to both boundaries. The question of interest is the evolution of their dispersion law with distance between boundaries. Then at low energies the entire system will consists of pairs of boundary states so that the spectrum and density of states for the entire system can be computed from the spectrum of a single pair.

To model a pair we choose c_n in the form

$$c_n = \begin{cases} -\gamma n_0 d \tanh(2(n-n_1)/n_1), n > 0 \\ \gamma n_0 d \tanh(2(n+n_1)/n_1), n < 0 \end{cases} \tag{28}$$

where n_1 determines the position of the boundaries. Since each boundary line goes through the lattice sites, the wave function on the left side is independent from the wave function on the right side for each boundary. This means that there exists a state "locked" between boundaries. The example of such a state is shown in Fig. 6. Contrary to the hybridization of states in two potential wells, these states are neither symmetric nor antisymmetric.

The energy spectrum of a pair can be computed using (22) the same way as above. In contrary to a single boundary case, there appears to be a finite gap at small energies. The width of the gap depends on the distance between boundaries and decreases with increasing the distance. The gap in the spectrum shows that the density of states vanishes at zero energy.

The crucial question is, can a spectrum of a pair for the varying distance between boundaries be described by a universal formula? It is obvious that a large k_y the spectrum should have the shape given by Eq. (17), though at small k_y it should saturate. The crossover value is $k_y \sim 1/n_1 d$. Thus, if the universal expression exists, it should have the following form

$$|E| = \frac{g_1}{\ln(g_2 F(kn_1 d)/kd)}. \tag{29}$$

where F is some universal function and $g_1/n = c_1$, $g_2 = c_2$ comes from the single boundary case. The numerical computation should be carried in the following way. If F exists, then all the branches for different distances between boundaries should collapse on the same curve after the proper rescaling of the axis. We present such a function in Fig. 7. For comparison, we also show Fig. 8, where scaling is missing.

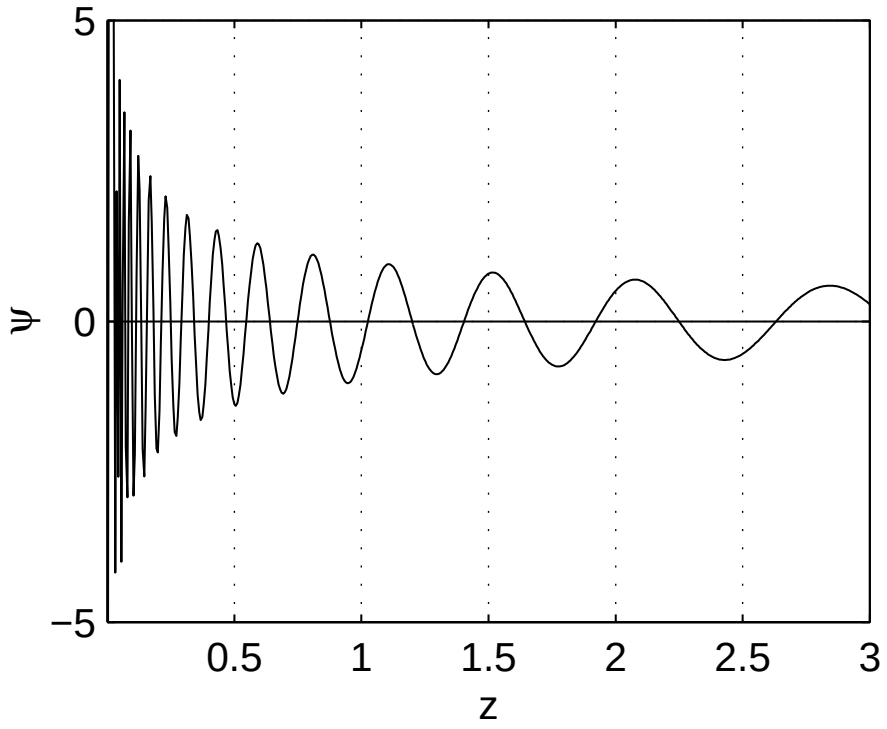

FIG. 1. Wave function for small values of z

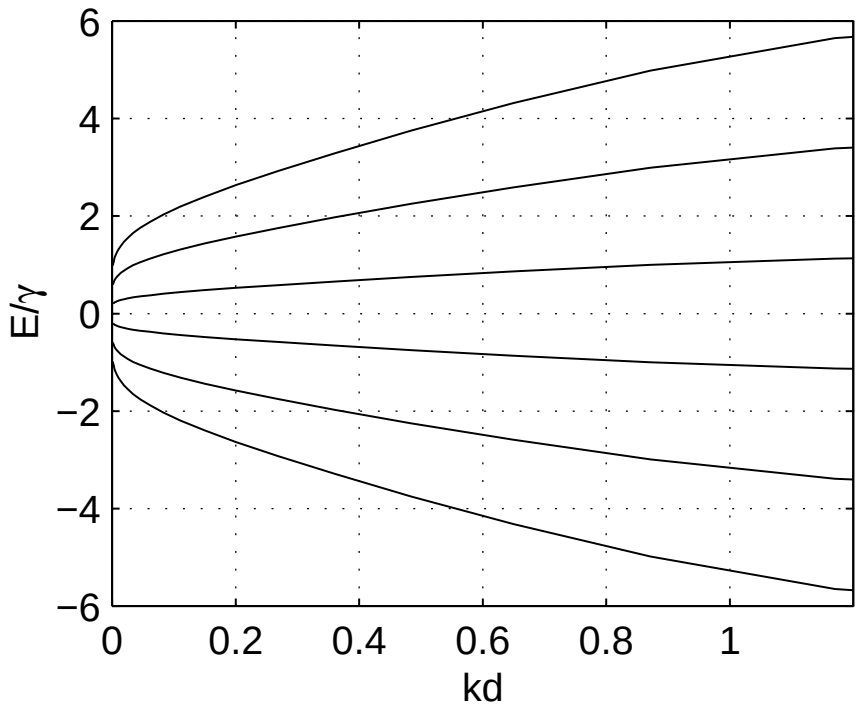

FIG. 2. The three lowest branches of energy spectrum for the eigenstates ,$\alpha = \gamma x$

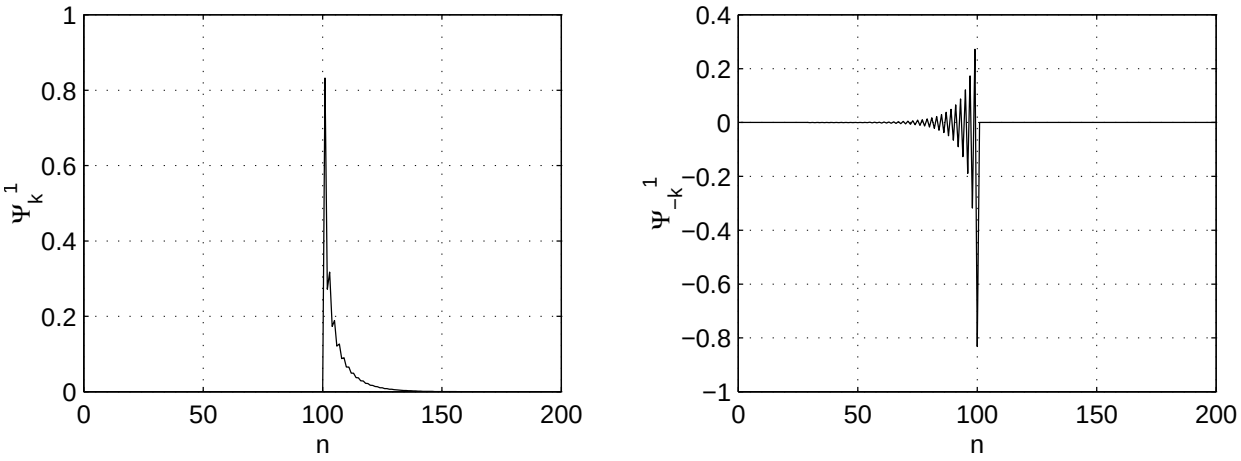

FIG. 3. Eigenfunctions of the lowest energy level for k_y and $-k_y$

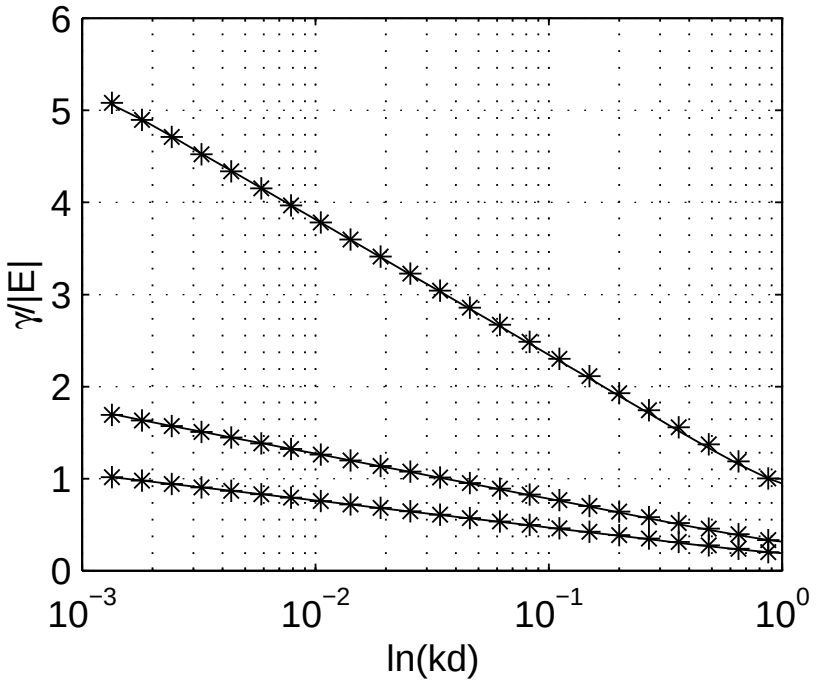

FIG. 4. Numerical fit to the asymptotic formula : solid line - asymptotics,stars - numerical values

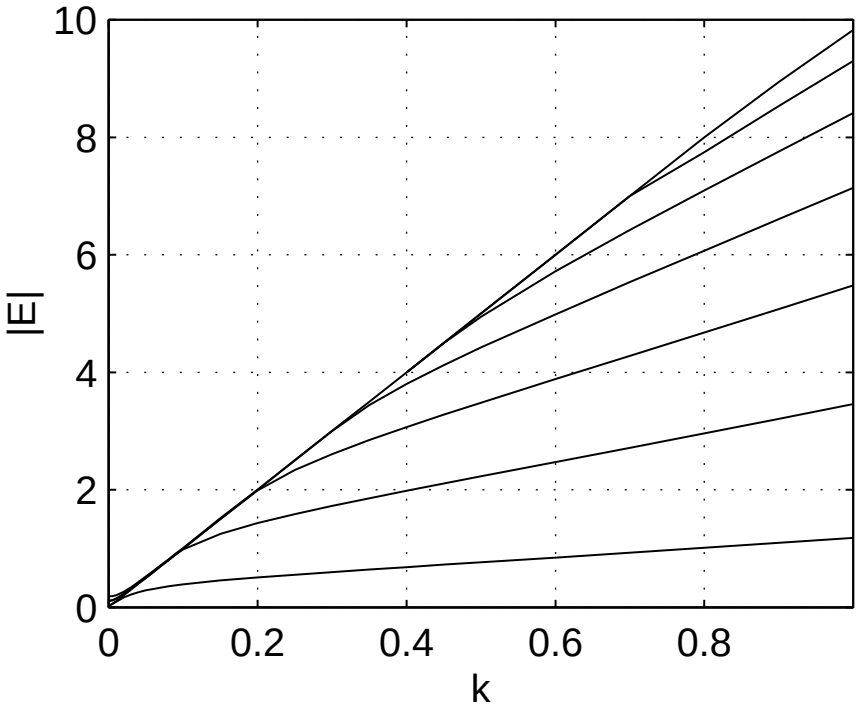

FIG. 5. Low energy part of the spectrum for $\alpha = \gamma tanh(x/x_0)$. Discrete spectrum on this figure coexist with continuum. The separation line is $E_c = \gamma n_0 k_y d$

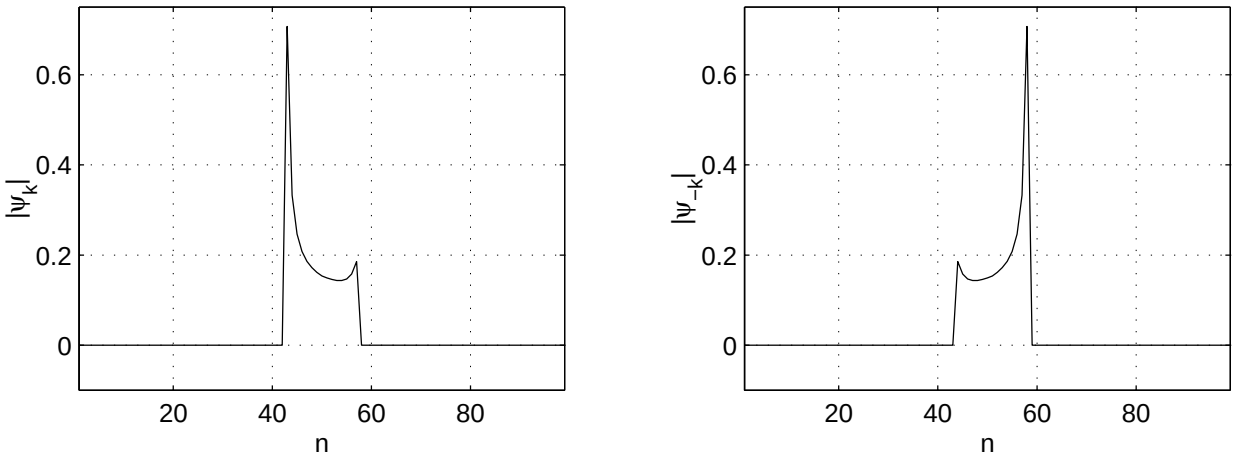

FIG. 6. Wave function of a pair of boundaries for k_y and $-k_y$

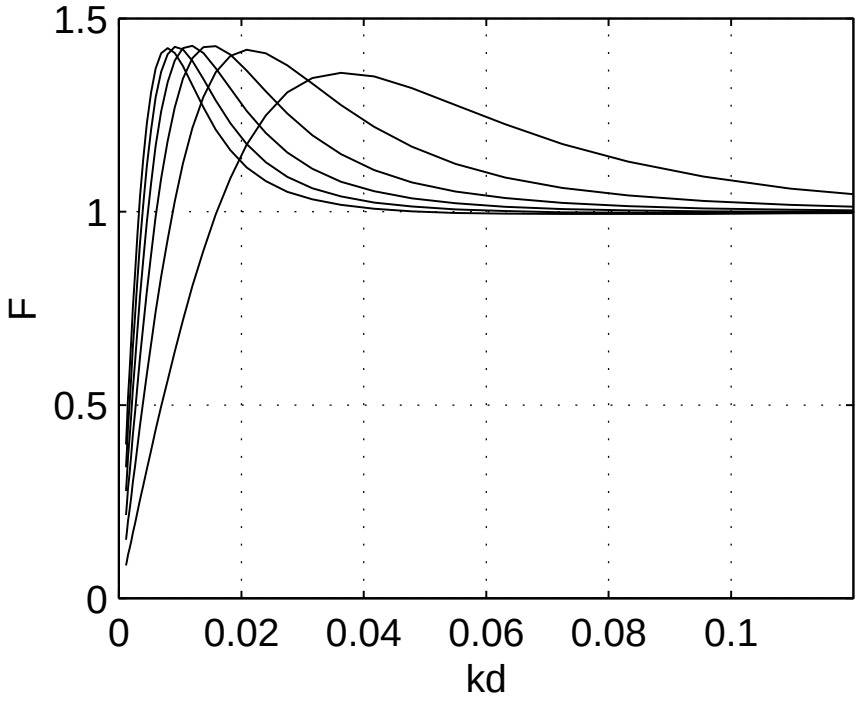

FIG. 7. Scaling is missing in this picture

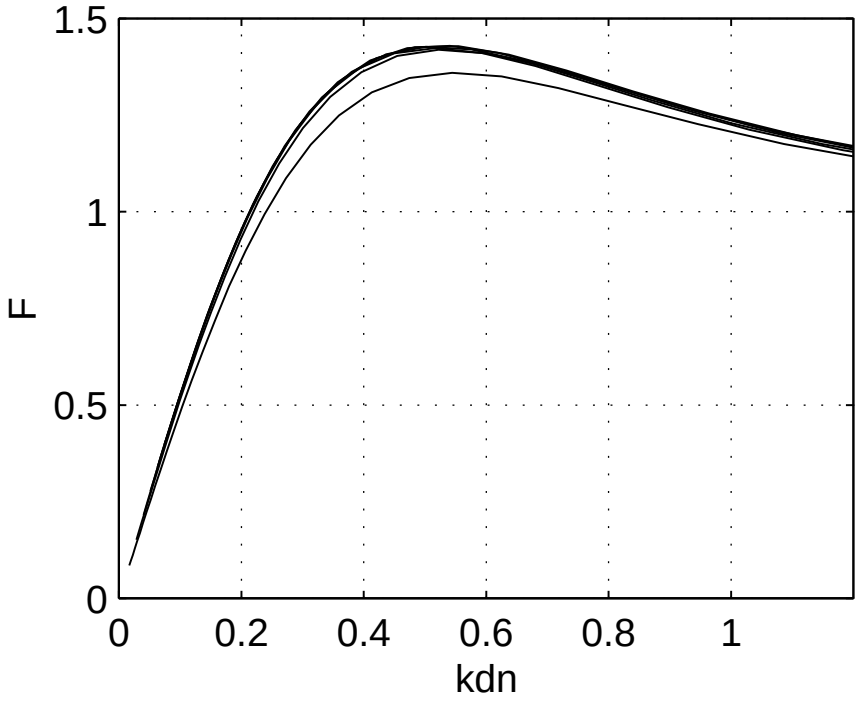

FIG. 8. Universal scaling function F from Eq.

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